Lower bounds for width-restricted clause learning

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Banff, 23. 01. 2014

partially based on joint work with

Sam Buss, Jan Hoffmann & Eli Ben-Sasson

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  $C_v \in F$ or $C_v = C_u$ for some $u \prec v$ (lemma)

$\prec$ is the post-order on trees.

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Here: lower bounds for $RTL(k)$:

A refutation R in RTL is in $RTL(k)$, if every lemma C used in R is of width $w(C) \leq k$.

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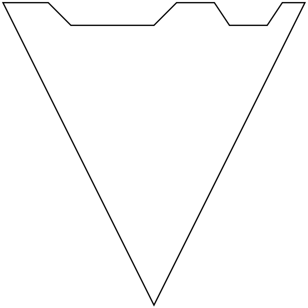

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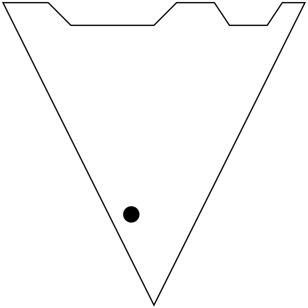

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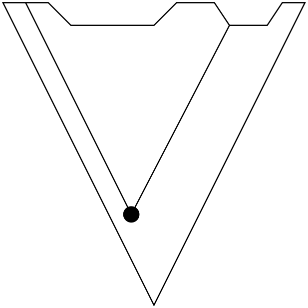

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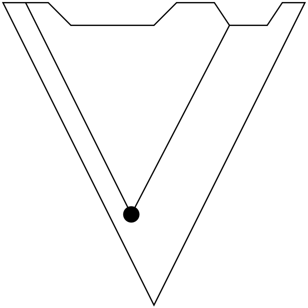

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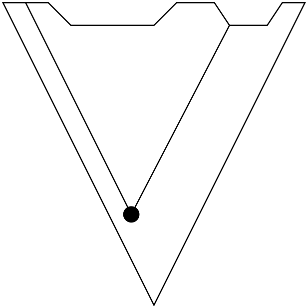

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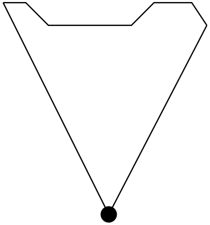

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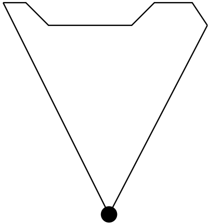

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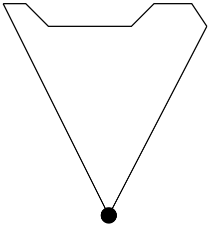

Main Lemma: For C in R with $w(C) \leq k$, there is a matching restriction ρ with $C\lceil\rho = 0$ and $|\rho| \leq k$

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- maximum clauses $\bigvee_{j \neq i} x_{i,j}$ for all i

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For clause C, the graph $G(C)$ has edges

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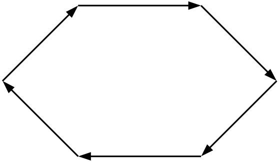

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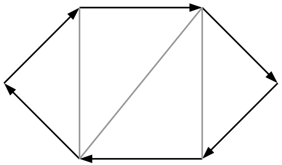

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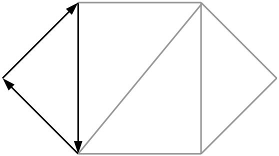

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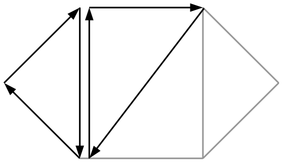

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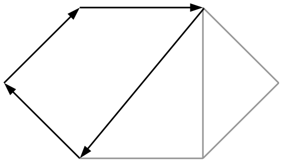

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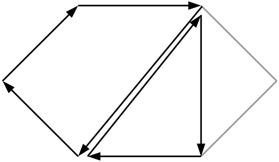

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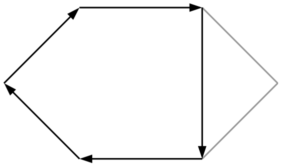

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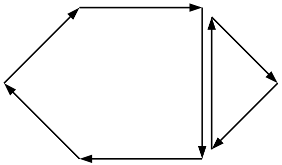

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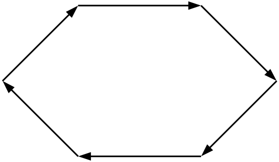

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$\rightsquigarrow$ obtain σ as a topological ordering of $G(C)$.

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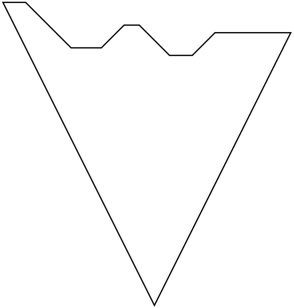

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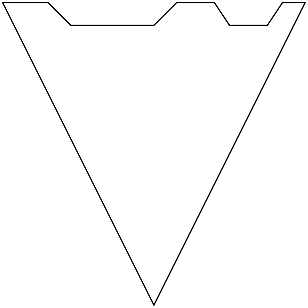

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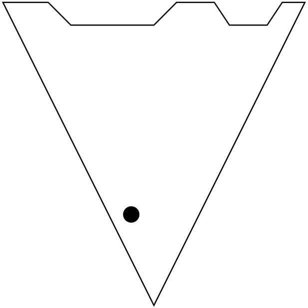

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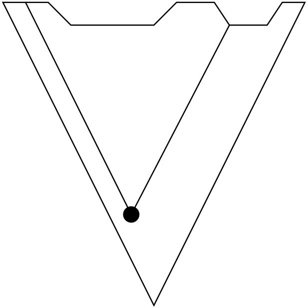

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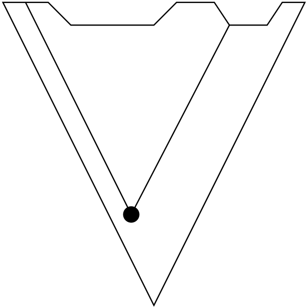

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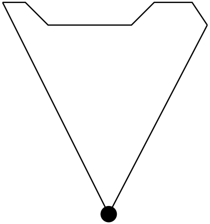

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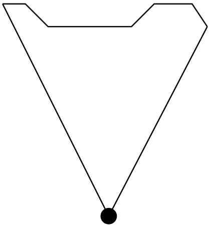

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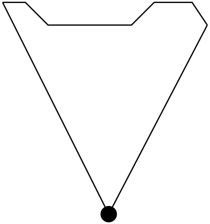

- Let R be a refutation of Ord_n
- Remove cyclic lemmas
- Find first C with $w(C) \leq k$
- Subtree R_C is tree-like derivation of C
- Pick σ with $C\lceil\sigma = 0$
- $R_C\lceil\sigma$ is refutation of $Ord_n\lceil\sigma$
- $Ord_n\lceil\sigma = Ord_{n-|\sigma|+1}$
- lower bound by BONET/GALESI

A Game

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$\mathcal{H}$ avoids F if $\mathcal{H}$ avoids all $C \in F$

Resolution width and systems of restrictions

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Theorem (Ben-Sasson & Wigderson)

If a d-CNF formula F requires resolution width w, then tree-like resolution proofs of F require size 2^{w-d}.

Restricted systems

Lemma

Let $\mathcal{H}$ be a w-system of restrictions over X, and $\rho \in \mathcal{H}$.

$$\mathcal{H}\lceil\rho := \{\, \sigma \,;\, \operatorname{dom}\sigma \subseteq X \setminus \operatorname{dom}\rho \;\textit{ and }\; \sigma \cup \rho \in \mathcal{H} \;\textit{ and }\; |\sigma| \leq w - |\rho| \,\}$$

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Lemma

If $\mathcal{H}$ avoids F, then $\mathcal{H}\lceil\rho$ avoids $F\lceil\rho$.

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- $R_C\lceil\rho$ is of size 2^{w-2k} by Ben-Sasson & Wigderson

Application

$E_3(F) := $ 3-CNF expansion of F

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This even holds for $k = k(n)$ *when* $k(n) = O(\log n)$.