Entropy production inequality for the linear Boltzmann equation

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The bilinear Boltzmann operator

For f, g functions on $\mathbb{R}^d$ we define

$$\mathcal{Q}(f,g)(v) = \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} B(|q|, \xi) \left(f(v')g(v'_*) - f(v)g(v_*) \right) \, \mathrm{d}v_* \, \mathrm{d}n.$$

where

$$\left.\begin{aligned} v' &= v - (q \cdot n)n \\ v'_* &= v_* + (q \cdot n)n \end{aligned}\right\} \quad \text{precollisional velocities}$$

n — direction of change of velocity

$q = v - v_*$ — relative velocity

$\xi = |q \cdot n| / |q|$ — cos of angle between q and n

The linear Boltzmann equation

We consider

$$\partial_t f = \mathcal{Q}(f, M)$$

where M is the Maxwellian

$$M(v) = (2\pi)^{-d/2} \exp\left(-\frac{|v|^2}{2}\right), \qquad v \in \mathbb{R}^d.$$

- This is a linear equation.
- It conserves mass and positivity.
- M is an equilibrium, since $\mathcal{Q}(M, M) = 0$.

This linear Boltzmann equation has many Lyapunov functionals:

$$H_\Phi(f|M) = \int_{\mathbb{R}^d} \mathcal{M}(v)\Phi\left(\frac{f(v)}{\mathcal{M}(v)}\right) \mathrm{d}v$$

for any convex Φ. A particular example for $\Phi(x) = x \log x$ is the *Shannon-Boltzmann relative entropy*,

$$H(f|M) = \int_{\mathbb{R}^d} f(v) \log\left(\frac{f(v)}{\mathcal{M}(v)}\right) \mathrm{d}v.$$

We have

$$\frac{\mathrm{d}}{\mathrm{d}t} H(f|M) = -D(f)$$

where

$$D(f) := \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} B\,\mathcal{M}(v)\mathcal{M}(v_*)\Psi\left(\frac{f(v)}{\mathcal{M}(v)}, \frac{f(v')}{\mathcal{M}(v')}\right) \mathrm{d}n\,\mathrm{d}v_*\,\mathrm{d}v,$$

with $\Psi(x,y) := (x-y)(\log x - \log y) \geq 0$.

The question we address is the following: can one prove a functional inequality

$$\lambda H(f|M) \leq D(f)$$

between the relative entropy and its production, for all probability distributions f?

- This is a well-known technique to prove exponential convergence to equilibrium.
- In particular, the corresponding inequality for the *nonlinear* Boltzmann equation is known as *Cercignani's conjecture*. It is a well-known problem, studied by many people [Bobylev, Cercignani, Desvillettes, Toscani, Villani...]
- This is also part of a general kind of inequalities known as "modified log-Sobolev inequalities" in probability.

Entropy dissipation inequalities

Take a linear operator with kernel $k(x,y)$ and equilibrium M.

SPECTRAL GAP:

$$\lambda_2 \int M \left(\frac{f}{M} - 1 \right)^2 \leq \iint k(x,y) M(x) \left(\frac{f(x)}{M(x)} - \frac{f(y)}{M(y)} \right)^2$$

MODIFIED LOG-SOBOLEV / ENTROPY-ENTROPY PRODUCTION:

$$\lambda_0 \int f \log \frac{f}{M} \leq \iint k(x,y) M(x) \Psi \left(\frac{f(x)}{M(x)}, \frac{f(y)}{M(y)} \right)$$

with $\Psi(x,y) := (x-y)(\log x - \log y)$.

LOGARITHMIC SOBOLEV:

$$\lambda \int f \log \frac{f}{M} \leq \iint k(x,y) M(x) \left(\sqrt{\frac{f(x)}{M(x)}} - \sqrt{\frac{f(y)}{M(y)}} \right)^2$$

Let B be the hard-spheres collision kernel

$$B(|q|,\xi) = |q \cdot n| = |q|\xi,$$

or a Maxwellian collision kernel,

$$B(|q|,\xi) = b(\xi).$$

Theorem (Bisi, C., Lods)

There exists $\lambda_0 > 0$ such that

$$\lambda_0 H(f|M) \leq D(f)$$

for all probability distributions f. More precisely, for Maxwellian molecules we can take

$$\lambda_0 = \gamma_b := \int_{\mathbb{S}^{d-1}} (\tilde{q} \cdot n)^2 b(\tilde{q} \cdot n)\, \mathrm{d}n$$

Sketch of proof

1. We first prove the inequality for Maxwellian collision kernels.
2. We then prove that the entropy production for hard-spheres is larger than a constant times that for Maxwell molecules. (Actually, the comparison is more general than that.)

First part: Maxwellian collision kernels

In the Maxwellian case we have

$$\mathcal{Q}(f, M) = \mathcal{Q}_+(f, M) - f$$

so

$$\begin{aligned}
\frac{d}{dt} H(f|M) &= \int_{\mathbb{R}^d} Q_+(f, M) \log \frac{f}{M} - H(f|M) \\
&= \int_{\mathbb{R}^d} Q_+(f, M) \log \frac{Q_+(f, M)}{M} \\
&\quad - \int_{\mathbb{R}^d} Q_+(f, M) \log \frac{Q_+(f, M)}{f} - H(f|M) \\
&= H(Q_+(f, M)|M) - H(Q_+(f, M)|f) - H(f|M)
\end{aligned}$$

Hence it is enough to show that

$$H(Q_+(f, M)|M) \leq (1 - \gamma_b) H(f|M)$$

Denote by H the entropy

$$H(f) := \int_{\mathbb{R}^d} f \log f.$$

We use the following inequality [Villani '98, Matthes & Toscani '12] for the positive part of the Boltzmann operator:

$$H(\mathcal{Q}_+(f,g)) \leq (1-\gamma_b)H(f) + \gamma_b H(g) \tag{1}$$

to write:

$$\begin{aligned} H(Q_+(f,M)|M) =& H(Q_+(f,M)) - \int_{\mathbb{R}^d} Q_+(f,M)\log M \\ \leq& (1-\gamma_b)H(f) + \gamma_b H(M) - \int_{\mathbb{R}^d} Q_+(f,M)\log M \\ =& (1-\gamma_b)H(f|M) \\ & {\color{red} -\gamma_b \int_{\mathbb{R}^d} (f-M)\log M - \int_{\mathbb{R}^d} Q(f,M)\log M} \end{aligned}$$

The term in red,

$$-\gamma_b \int_{\mathbb{R}^d} (f - M) \log M - \int_{\mathbb{R}^d} Q(f, M) \log M$$

is actually equal to 0, as can be calculated by looking at the explicit evolution of the energy (second moment) for the linear Boltzmann equation with Maxwellian collision kernel!

That proves the inequality for Maxwellian kernels. ■

Comparison between kernels

We also show the following inequality:

$$D_{\mathrm{hs}}(f) \geq C D_{\max}(f) \tag{2}$$

for all distributions f. Here D_{hs} and $D_{\max}$ are entropy productions for hard-sphere and Maxwellian kernels, respectively.
Actually, the functional D can be written as

$$D(f) := \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathcal{M}(v) k_B(v, v') \Psi \left(\frac{f(v)}{\mathcal{M}(v)}, \frac{f(v')}{\mathcal{M}(v')} \right) \mathrm{d}v \, \mathrm{d}v',$$

where k_B is the kernel of the linear operator $f \mapsto Q(f, M)$. Then (2) is actually deduced from a comparison of the operator kernels k for hard-spheres and Maxwellian molecules.

Comparison between kernels

We may use Carleman's change of variables to write

$$k_B(v', v) = \frac{1}{|v - v'|^{d-1}} \int_{E_{v,v'}} B(|q|, \xi) M(v'_*) \, \mathrm{d}v'_*, \qquad v', v \in \mathbb{R}^d$$

where now

$$E_{v,v'} \equiv \text{hyperplane through } v\text{, perpendicular to } v - v'$$

$$|q| = |2v - v' - v'_*|, \qquad |q \cdot n| = |v - v'|, \qquad \xi = \frac{|v - v'|}{|2v - v' - v'_*|}.$$

This expression for k_B can be compared for different kernels B.

Relation to log-Sobolev inequalities

If we consider the *grazing collisions* kernel

$$B(|q|, \xi) := \frac{\xi}{2\pi\epsilon} \chi_{[0,\epsilon]}(\xi)$$

for ϵ small, then the linear Boltzmann equation (with an appropriate time rescaling) is close to the Fokker-Planck equation

$$\partial_t f = \nabla \cdot \Big(D(v) \left(\nabla f + vf \right) \Big)$$

for a certain (explicit) diffusion matrix D.

When taking the limit $\epsilon \to 0$, our inequality gives a logarithmic Sobolev inequality for the limiting Fokker-Planck equation.

Thanks!