Critical exponents for degenerate Keller-Segel system

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Keller-Segel system, background

Patlak-Keller-Segel system: the basic mathematical model for Chemotaxis. It is a PDE model to describe the movement of cells in response to a chemical signal.

$$\rho_t = \Delta\rho - \mathrm{div}(\rho\nabla c), \quad -\Delta c = \rho.$$

Typical quantities of the system

- Conservation of mass

$$\boxed{m_0(t) = \int \rho(x,t)dx = \int \rho_0(x)dx = m_0.}$$

- Entropy (Lyapunov functional) dissipation relation,

$$\text{Entropy: } F(\rho) = \int (\rho \ln \rho - \frac{\rho c}{2})dx,$$

$$\boxed{\frac{d}{dt}F(\rho) + \int \rho|\nabla \ln \rho - \nabla c|^2 dx = 0.} \Rightarrow \boxed{F(\rho) \leq F(\rho_0)}$$

Key feature of the system: Global existence vs. finite time blow up. Since 1990's, Jäger and Luckhaus, Biler, Herrero, Horstmann, Medina, Nagai, Stevens, Velazquez, Winkler......

The critical mass 8π in 2-D

$$c = -\frac{1}{2\pi}\int \log\frac{1}{|x-y|}\rho(y)dy$$

Entropy: $F(\rho) = \int \rho \ln \rho dx - \frac{1}{8\pi}\int\int \rho(x,t)\rho(y,t)\log\frac{1}{|x-y|^2}dxdy.$

FACT 1: Entropy dissipation relation

$$\boxed{F(\rho) \le F(\rho_0).}$$

FACT 2: Logarithmic Hardy-Littlewood-Sobolev inequality

If $\rho \ge 0$ in L^1 and $\rho\log\rho \in L^1$, then

$$\boxed{\int \rho\log\rho dx - \frac{1}{m_0}\int\int \rho(x)\rho(y)\log\frac{1}{|x-y|^2}dxdy + C(m_0) \ge 0,}$$

where $m_0 = \int \rho(x)dx$, $C(m_0) := m_0(1 + \log\pi - \log m_0)$.

A direct application is that

$$F(\rho(\cdot,t)) \geq (1-\frac{m^0}{8\pi})\int \rho\log\rho dx - \frac{m^0}{8\pi}C(m^0),$$
$$F(\rho(\cdot,t)) \geq (\frac{1}{m^0}-\frac{1}{8\pi})\int\int \rho(x,t)\rho(y,t)\log\frac{1}{|x-y|^2}dxdy - C(m^0).$$

$m_0 < 8\pi$: **global existence**, Blanchet, Dolbeault, Perthame, 2006.
Another proof: Carrillo, Chen, Liu, and Wang, based on Delort's theory of 2-D incompressible Euler equation, 2012.

$m_0 > 8\pi$: **blow up in finite time**, Dolbeault, Perthame, 2004.
Idea: Second moment, $m_2(t) := \int \frac{|x|^2}{2}\rho dx$. $m_2'(t) = 2m_0(1-\frac{m_0}{8\pi}) < 0$.
$m_0 = 8\pi$: Blanchet, Carlen, Carrillo, Masmoudi.

Diffusion $\sim$ **Aggregation** in 2-D.

Multi-D and the choice of critical exponent

How about Multi-D ($n \geq 3$)?

To balance the aggregation, (Diffusion coefficient depends on the density itself)

$$\boxed{\text{Diffusion } -\Delta\rho} \rightsquigarrow \boxed{\text{Nonlinear Diffusion } -\Delta\rho^m}$$

Degenerate Keller-Segel in Multi-D

$$\boxed{\rho_t = \Delta\rho^m - \mathrm{div}(\rho\nabla c) = \mathrm{div}(\rho\nabla(\tfrac{m}{m-1}\rho^{m-1} - c)), x \in \mathbb{R}^n,\ \ t \geq 0.}$$

A known exponent, $m^* = 2 - \frac{2}{n}$ is chosen by a scaling invariance of the total mass. (if $\rho(x,t)$ is a solution, then $\lambda^n\rho(\lambda x, t)$ is also a solution)

Many results for $m = m^*$ since 2005: Bedrossian, Bertozzi, Blanchet, Carrillo, Cieślak, Horstmann, Ishida, Kowalczyk, Laurencot, Luckhaus, Rodríguez, Sugiyama, Szymańska ,Winkler, Yao, Yokota ...

What happens for $m = m^*$?

Answer: No nontrivial positive stationary solution for vanishing chemical potential!

- Nonnegative stationary solution with compact support, Blanchet, Carrillo, Laurencot, 2008.
- For more general exponent m, detailed discussions on stationary solution with compact support, Bian and Liu, 2013.
- $2-d$, compactly supported stationary solution, Carrillo, Castorina and Volzone, 2014.

Stationary solutions in 2-D, $\ln \rho - c = 0$,

$$-\Delta c = e^c \text{ in } \mathbb{R}^2,$$

has a family of solutions $C_{\lambda,x^0}(x)$, and the stationary solution for ρ is

$$U_{\lambda,x^0}(x) = e^{C_{\lambda,x^0}(x)} = 8\left(\frac{\lambda}{\lambda^2 + |x - x_0|^2}\right)^2, \text{ with } \|U_{\lambda,x^0}\|_{L^1} = 8\pi.$$

Stationary solutions in Multi-D, $\frac{m}{m-1}\rho^{m-1} - c = 0$,

$$-\Delta c = \left(\frac{m-1}{m}\right)^{\frac{1}{m-1}} c^{\frac{1}{m-1}} \text{ in } \mathbb{R}^n.$$

Gidas, Spruck in 1981 proved

$1 \le \frac{1}{m-1} < \frac{n+2}{n-2}$, then $c \equiv 0$. m^* belongs to this case.

$\frac{1}{m-1} = \frac{n+2}{n-2}$, or $m = \frac{2n}{n+2}$, then $c = C_{\lambda,x_0}(x)$, and the stationary solution is

$$U_{\lambda,x_0}(x) = 2^{\frac{n+2}{4}} n^{\frac{n+2}{2}} \left(\frac{\lambda}{\lambda^2 + |x - x_0|^2}\right)^{\frac{n+2}{2}} \text{ with } \|U_{\lambda,x_0}\|^m_{L^m} = K(n).$$

New exponent $m_c = 2n/(n+2)$.

- There exists a family of positive stationary solutions.
- The entropy is conformal invariant.
- There is a clear criteria to classify initial data for either global existence or blow up.
- ……

Radially symmetric solutions

Theorem (CI, J.-G. Liu, J. Wang, SIAM Math. Anal., 2012)

For radially symmetric $\rho_0 \geq 0$.

1. *If* $\boxed{\rho_0(|x|) < U_{\lambda_0}(|x|)}$ *for some* $\lambda_0 > 0$, *then*

$$\rho(|x|, t) \to 0, \quad \text{in } L^1_{loc}(\mathbb{R}^n) \text{ as } t \to \infty.$$

2. *If* $\boxed{\rho_0(|x|) > U_{\lambda_0}(|x|)}$ *for some* $\lambda_0 > 0$, *then* $\rho(|x|, t)$ *must blow up at a time* $t^* \leq +\infty$, *i.e.,* $\exists r(t) \to 0$ *as* $t \to t^*$ *and a constant* $C > 0$ *s.t.*

$$\int_{B(0,r(t))} \rho(|x|, t) dx \geq C.$$

Remark The assumptions can be reformulated in a slightly general version, for example, like in Kim and Yao's work;

Remark The stationary solution is not stable.

Back to general case: $m_c = \frac{2n}{n+2}$

$$\begin{aligned} \rho_t &= \Delta \rho^{m_c} - \mathrm{div}(\rho \nabla c), && x \in \mathbb{R}^n,\ t \geq 0, \\ -\Delta c &= \rho, && x \in \mathbb{R}^n,\ t \geq 0, \\ \rho(x,0) &= \rho_0(x), && x \in \mathbb{R}^n \end{aligned}$$

Entropy

$$\mathcal{F}(\rho) = \frac{1}{m_c - 1} \int \rho^{m_c}(x,t)dx - \frac{c_n}{2} \int \int \frac{\rho(x,t)\rho(y,t)}{|x-y|^{n-2}} dxdy.$$

Energy dissipation relation

$$\frac{d\mathcal{F}(\rho)}{dt} + \int \rho \left| \nabla \left(\frac{m_c}{m_c - 1} \rho^{m_c - 1} - c \right) \right|^2 dx = 0.$$

Existence and blow up

Theorem (Cl, J.-G. Liu, J. Wang, SIAM Math. Anal., 2012)

$\rho_0 \in L^1_+ \cap L^{m_c}$, $\boxed{\|\rho_0\|_{L^{m_c}} < C_s < \|U_\lambda\|_{L^{m_c}}}$, $\Rightarrow$ **GLOBAL WEAK SOLUTION**. *Moreover, for large t, it holds*

$$\|\rho(\cdot,t)\|_{L^{m_c}} \le Ct^{-\frac{1}{m_c(\beta-1)}}, \quad \beta = \frac{2m_c^2 - 3m_c + 2}{m_c(m_c-1)} > 1.$$

Idea L^{m_c} estimates and compactness argument.

Theorem (Cl, J.-G. Liu, J. Wang, SIAM Math. Anal., 2012)

$m_2(0) < \infty$, $\mathcal{F}(\rho_0) < \mathcal{F}(U_\lambda)$ *and* $\boxed{\|\rho_0\|_{L^{m_c}} > \|U_\lambda\|_{L^{m_c}}}$, $\Rightarrow$ **SOLUTION BLOW UP** *in the sense that* $\exists T^* < \infty$ *s.t.*

$$\lim_{t\to T^*} \|\rho(\cdot,t)\|_{L^{m_c}} = +\infty.$$

Idea Entropy decomposition and some new ideas.

Remark L^m blow up implies L^∞ blow up, due to conservation of mass.

How to understand m^* and m_c? $\dfrac{4}{3}$ or $\dfrac{6}{5}$?

What happens for $\frac{2n}{n+2} = m_c < m < m^* = 2 - \frac{2}{n}$?

$$\begin{aligned}
&\rho_t = \Delta\rho^m - \mathrm{div}(\rho\nabla c), && x \in \mathbb{R}^n,\ t \geq 0, \\
&-\Delta c = \rho, && x \in \mathbb{R}^n,\ t \geq 0, \qquad m \in \left(\tfrac{2n}{n+2}, 2 - \tfrac{2}{n}\right) \\
&\rho(x,0) = \rho_0(x), && x \in \mathbb{R}^n,
\end{aligned}$$

Theorem (CI, J.Wang, Documenta Math. 2014)

$\rho_0 \in L^1_+(\mathbb{R}^n) \cap L^{\frac{2n}{n+2}}(\mathbb{R}^n)$, $\mathcal{F}(\rho_0) < \mathcal{F}^*$, *the following holds,*

1. $\|\rho_0\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} < (s^*)^{\frac{n-2}{2n(m-1)}}$ **GLOBAL WEAK SOLUTION**
2. $\|\rho_0\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} > (s^*)^{\frac{n-2}{2n(m-1)}}$ $m_2(0) < \infty$, **FINITE TIME BLOW UP**.

where $\mathcal{F}^* = K_1(n,m) m_0^{\alpha_1(m,n)} > 0, \quad s^* = K_2(n,m) m_0^{\alpha_2(m,n)}$.

REMARK: If $\mathcal{F}(\rho_0) < \mathcal{F}^*$, $L^{\frac{2n}{n+2}}$ norm of the initial data can not be $(s^*)^{\frac{n-2}{2n(m-1)}}$, which can be easily checked by using the decomposition of the free energy. Thus the classification of the initial data is complete.

REMARK: The result does not hold for $m = m^* = 2 - \frac{2}{n}$, thus there is no contradiction with the result by Blanchet *et al.*, where a critical mass was obtained.

REMARK:

$$\|\rho_0\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} < (s^*)^{\frac{n-2}{2n(m-1)}}, \Rightarrow \mathcal{F}(\rho_0) > 0.$$

$$\mathcal{F}(\rho_0) < 0 \Rightarrow \|\rho_0\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} > (s^*)^{\frac{n-2}{2n(m-1)}}.$$

Therefore, our result on the blow-up of solutions allows more initial data than those in the work by Sugiyama.

REMARK: The assumption $\mathcal{F}(\rho_0) < \mathcal{F}^*$ in Theorem gives a relation between the initial mass and the initial free energy, i.e.

$$\mathcal{F}(\rho_0) M_0^{\frac{m(n+2)-2n}{2n-2-mn}} < \frac{2-\frac{2}{n}-m}{(m-1)(1-\frac{2}{n})} \left(\frac{2n^2\alpha(n)}{C(n)}\right)^{\frac{n(m-1)}{2n-2-mn}}. \quad (1)$$

As a conclusion, *the initial mass itself might not be an important quantity in the existence and blow up analysis in the intermediate exponent*. More precisely, no matter how small the initial mass is, the solution can still blow up in case that $\|\rho_0\|_{L^{\frac{2n}{n+2}}} > (s^*)^{\frac{n-2}{2n(m-1)}}$. No matter how large the initial mass is, there still exists a global weak solution if $\|\rho_0\|_{L^{\frac{2n}{n+2}}} < (s^*)^{\frac{n-2}{2n(m-1)}}$.

Idea (for $m_c < m < m^*$)

$$\begin{aligned}
\mathcal{F}(\rho) &= \frac{1}{m-1}\int_{\mathbb{R}^n} \rho^m(x,t)dx - C(n)\tilde{C}\|\rho\|^2_{L^{\frac{2n}{n+2}}} \\
&\quad + C(n)\tilde{C}\|\rho\|^2_{L^{\frac{2n}{n+2}}} - \tilde{C}\int\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\rho(x,t)\rho(y,t)}{|x-y|^{n-2}}dxdy. \\
&=: \mathcal{F}_1(\rho) + \mathcal{F}_2(\rho).
\end{aligned}$$

HLS inequality implies that $\mathcal{F}_2(\rho) \geq 0$.

Interpolation $\boxed{\|\rho\|_{L^{\frac{2n}{n+2}}} \leq \|\rho\|^{1-\theta}_{L^1}\|\rho\|^{\theta}_{L^m},\ \theta = \frac{m(n-2)}{2n(m-1)}}$ gives,

$$\begin{aligned}
\mathcal{F}_1(\rho) &= \frac{1}{m-1}\int_{\mathbb{R}^n} \rho^m(x,t)dx - C(n)\tilde{C}\|\rho\|^2_{L^{\frac{2n}{n+2}}} \\
&\geq \frac{1}{m-1}M_0^{\frac{2n-m(n+2)}{n-2}}\|\rho\|^{\frac{2n(m-1)}{n-2}}_{L^{\frac{2n}{n+2}}} - C(n)\tilde{C}\|\rho\|^2_{L^{\frac{2n}{n+2}}}.
\end{aligned}$$

Let

$$f(s) = \frac{1}{m-1} M_0^{\frac{2n-m(n+2)}{n-2}} s - \frac{C(n)}{2(n-2)n\alpha(n)} s^{\frac{n-2}{n(m-1)}}.$$

$f(s)$ is a concave function with maximum point s^*.

Therefore, there exists a lower bound of the first part of free energy

$$f\left(\|\rho\|_{L^{\frac{2n}{n+2}}}^{\frac{2n(m-1)}{n-2}} \right) \leq \mathcal{F}_1(\rho).$$

If $\mathcal{F}(\rho_0) < \mathcal{F}^* := f(s^*)$, we have the sequence of inequalities,

$$f\left(\|\rho\|_{L^{\frac{2n}{n+2}}}^{\frac{2n(m-1)}{n-2}} \right) \leq \mathcal{F}_1(\rho) \leq \mathcal{F}(\rho) \leq \mathcal{F}(\rho_0) < \delta f(s^*).$$

If in addition, $\|\rho_0\|_{L^{\frac{2n}{n+2}}} > (s^*)^{\frac{n-2}{2n(m-1)}}$, then $\exists \mu_2 > 1$ such that

$$\|\rho(\cdot,t)\|_{\frac{2n}{n+2}} > (\mu_2 s^*)^{\frac{n-2}{2n(m-1)}}, \text{ for all } t > 0,$$

Second moment and blow up

$$
\begin{aligned}
\frac{dm_2(t)}{dt} &= \left(2n - \frac{2(n-2)}{m-1}\right) \int_{\mathbb{R}^n} \rho^m dx + 2(n-2)\mathcal{F}(\rho) \\
&\leq \left(2n - \frac{2(n-2)}{m-1}\right) M_0^{\frac{(\theta-1)m}{\theta}} \|\rho\|_{L^{\frac{2n}{n+2}}}^{\frac{m}{\theta}} + 2(n-2)\mathcal{F}(\rho_0) \\
&< \left(2n - \frac{2(n-2)}{m-1}\right) M_0^{\frac{(\theta-1)m}{\theta}} \mu_2 s^* + 2(n-2) f(s^*) \\
&= \left(2n - \frac{2(n-2)}{m-1}\right) M_0^{\frac{(\theta-1)m}{\theta}} (\mu_2 - 1) s^* < 0.
\end{aligned}
$$

$m_2'(t) < 0$ and $m_2(0) < +\infty$ implies "blow up".

$\forall R > 0$, by using Hölder inequality, we have

$$\int_{\mathbb{R}^n} \rho(x)dx \leq \int_{B_R} \rho(x)dx + \int_{B_R^c} \rho(x)dx \leq CR^{\frac{n-2}{2}} \|\rho\|_{L^{m_c}} + \frac{1}{R^2} m_2(t).$$

Now by choosing $R = (\frac{m_2(t)}{C\|\rho\|_{L^{m_c}}})^{2/(n+2)}$, we have

$$\|\rho\|_{L^1} \leq C\|\rho\|_{L^{m_c}}^{\frac{4}{n+2}} m_2(t)^{\frac{n-2}{n+2}}.$$

So,

$$\lim_{t \to T} \|\rho\|_{L^{m_c}}^{\frac{4}{n+2}} \geq \lim_{t \to T} \frac{\|\rho\|_{L^1}}{\bar{C}(n) m_2(t)^{\frac{n-2}{n+2}}} = \infty.$$

THANK YOU!