The Curvature-Dimension Condition with Finite N

Consequences and Transformations

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Two Possible Introductions

Equivalence of Bakry-Emery and Curvature-Dimension Conditions

Gradient Flows for \((K, N)\)-Convex Functions

Time Change and Conformal Transformation

Analysis and Geometry on Metric Measure Spaces
Two Possible Introductions

Heat flow on \((X, d, m)\) is the gradient flow for the entropy \(S\)

- Functional inequalities are determined by convexity bounds for \(S\) (\(=\) Ricci bounds for \(X\)): \(CD(K, \infty)\)-condition
- What do we gain if we assume the more restrictive \(CD(K, N)\)-condition with finite \(N\)?
- What can we say about the transformed space \((X, d', m')\) with \(m' = e^V m\) and \(d' \approx e^W d\) on infinitesimal scale?
Heat flow on \((X, d, m)\) is the gradient flow for the entropy \(S\)

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Heat flow is defined by the diffusion operator \(L\) on \(L^2(X, m)\)

- Functional inequalities are determined by the Bakry-Emery condition \(BE(K, \infty)\) for \(L\)
- What do we gain if we assume the more restrictive \(BE(K, N)\)-condition with finite \(N\)?
- What can we say about the transformed operators \(\tilde{L}u = Lu + \Gamma(V, u)\) or \(L'u = e^{-2W}Lu\)?
Given $\mu_0, \mu_1 \in \mathcal{P}_2(M)$ with $\mu_0 \ll m$. Then there exists a unique geodesic $(\mu_t)_{0 \leq t \leq 1}$ connecting them, given as

$$\mu_t := (F_t)_* \mu_0,$$

where

$$F_t(x) = \exp_x (t \nabla \varphi(x))$$

with suitable $d^2/2$-convex $\varphi : M \to \mathbb{R}$.

In the case $M = \mathbb{R}^n$ this states that there exists a convex function $\varphi_1$ such that

$$F_t(x) = x + t \nabla \varphi(x) = (1 - t)x + t \nabla \varphi_1(x).$$

**Tangent space:**

$$T_{\mu_0} \mathcal{P}_2 = \text{closure of } \{ \Phi = \nabla \varphi : M \to TM, \int_M |\nabla \varphi|^2 d\mu_0 < \infty \}$$
Consider $S : \mathcal{P}_2(M) \to \mathbb{R}$ with $S(\mu) = \int_M \rho \log \rho \, dm$ if $\mu = \rho \, m$.

The gradient $\nabla S(\mu) \in T_\mu \mathcal{P}_2(M)$ of $S$ at $\mu = \rho \, m$ is given by

$$\nabla S(\mu) = \nabla \log \rho.$$

The gradient flow $\frac{\partial \mu}{\partial t} = -\nabla S(\mu)$ on $\mathcal{P}_2(M)$ for $S$ is given by $\mu_t(dx) = \rho_t(x) m(dx)$ where $\rho$ solves the heat equation

$$\frac{\partial}{\partial t} \rho = \triangle \rho \quad \text{on } M.$$
$M$ complete Riemannian manifold, $m$ Riemannian volume measure

**Theorem.** (Otto '01, Otto/Villani '00, Cordero/McCann/Schmuckenschläger '01, vRenesse/Sturm '05)

Let $\text{Ent}(\mu) = \int \rho \cdot \log \rho \, dm$ with $\rho = \frac{d\mu}{dm}$. Then

$$\text{Hess } \text{Ent} \geq K \iff \text{Ric}_M \geq K$$

The **proof** depends on the following estimate for the logarithmic determinant $y_t := \log \det dF_t$ of the Jacobian of the transport map:

$$\ddot{y}_t(x) \leq -\frac{1}{n} (\dot{y})^2_t(x) - \text{Ric}(\dot{F}_t(x), \dot{F}_t(x))$$

This inequality is sharp. It describes the effect of curvature on optimal transportation.
Definition. $\text{CD}(K, \infty)$ or $\text{Ric}(X, d, m) \geq K$

$\iff \forall \mu_0, \mu_1 \in \mathcal{P}_2(X) : \exists$ geodesic $\mu_t$ s.t. $\forall t \in [0, 1]$:

$$\text{Ent} (\mu_t | m) \leq (1 - t) \text{Ent} (\mu_0 | m) + t \text{Ent} (\mu_1 | m) - \frac{K}{2} t(1 - t) W_2^2 (\mu_0, \mu_1)$$

Ent($\nu | m$) = \begin{cases} 
\int_X \rho \log \rho \, dm, & \text{if } \nu = \rho \cdot m \\
+ \infty, & \text{if } \nu \ll m
\end{cases}

$W_2(\mu_0, \mu_1) = \inf_q \left[ \int_{X \times X} d^2(x, y) \, d q(x, y) \right]^{1/2}$
The Curvature-Dimension Condition \( CD(0, N) \)

**Definition.** \( CD(0, N) \)

\[
\iff \forall \mu_0, \mu_1 \in \mathcal{P}_2(X) : \exists \text{ geodesic } (\mu_t)_{t} \text{ s.t. } \forall t \in [0, 1] : \\
S_N(\mu_t|m) \leq (1 - t)S_N(\mu_0|m) + t S_N(\mu_1|m) 
\]

Here \( S_N(\nu|m) = - \int_X \rho^{1-1/N} \, dm \) for \( \nu = \rho \cdot m + \nu_s \)

\[ \begin{align*}
\text{sec} \geq 0 & \iff \text{dist concave} \\
\text{ric} \geq 0 & \iff \text{vol}^{1/n} \text{ concave}
\end{align*} \]
**Definition.** (Bacher/St. JFA 2010) A metric measure space \((X, d, m)\) satisfies the ('reduced') **Curvature-Dimension Condition** \(CD^*(K, N)\) for \(K, N \in \mathbb{R}\), iff

\[
\forall \rho_0 m, \rho_1 m : \exists \text{ geodesic } \rho_t m \text{ and optimal coupling } q \text{ satisfying }
\]

\[
\int_X \rho_t^{1-1/N}(z) \, dm(z) \geq \int_{X \times X} \left[ \sigma_{K,N}^{(1-t)}(\gamma_0, \gamma_1) \cdot \rho_0^{-1/N}(\gamma_0) + \sigma_{K,N}^{(t)}(\gamma_0, \gamma_1) \cdot \rho_1^{-1/N}(\gamma_1) \right] \, dq(\gamma_0, \gamma_1)
\]

where \(\sigma_{K,N}^{(t)}(x, y) = \frac{\sin(\sqrt{\frac{K}{N}} t \, d(x,y))}{\sin(\sqrt{\frac{K}{N}} d(x,y))}\). In particular, \(\sigma_{0,N}^{(t)}(x, y) = t\).
The Curvature-Dimension Condition \( CD^*(K, N) \)

Riemannian manifolds:

\[ CD^*(K, N) \iff \text{Ric}_M \geq K \text{ and } \dim M \leq N \]

Weighted Riemannian spaces \((M, d, m)\) with \( dm = e^{-V} d\text{vol} \):

\[ \text{Ric}_M + \text{Hess} V - \frac{1}{N - n} D V \otimes D V \geq K \text{ and } \dim M \leq N \]

Further examples: Ricci limit spaces, Alexandrov spaces, Finsler manifolds (e.g. Banach spaces), Wiener space \((K = 1, N = \infty)\).

Constructions: Products, cones, suspensions, warped products.
Riemannian manifolds:

\[ CD^*(K, N) \iff \text{Ric}_M \geq K \quad \text{and} \quad \text{dim}_M \leq N \]

Weighted Riemannian spaces \((M, d, m)\) with \(dm = e^{-V}dvol\):

\[ \text{Ric}_M + \text{Hess}V - \frac{1}{N-n}DV \otimes DV \geq K \quad \text{and} \quad \text{dim}_M \leq N \]

Further examples: Ricci limit spaces, Alexandrov spaces, Finsler manifolds (e.g. Banach spaces), Wiener space \((K = 1, N = \infty)\).

Constructions: Products, cones, suspensions, warped products.

**Theorem (Ketterer 2014) "Cone Theorem"**

For any \(\kappa \geq 0\) and \(N \geq 1\) the following are equivalent:

- \((X, d, m)\) satisfies \(CD^*(N - 1, N)\) and has diameter \(\leq \pi\)
- The \((\kappa, N)\)-cone over \((X, d, m)\) satisfies \(CD^*(\kappa N, N + 1)\)

\(\kappa = 0\): Euclidean cone; \(\kappa = 1\): spherical suspension.
Heat equation on $X$

- either as gradient flow on $L^2(X, m)$ for the energy

$$
\mathcal{E}(u) = \frac{1}{2} \int_X |\nabla u|^2 \, dm = \lim\inf_{v \to u \text{ in } L^2} \frac{1}{2} \int_X (\text{lip}_x v)^2 \, dm(x)
$$

with $|\nabla u|$ = minimal weak upper gradient

- or as gradient flow on $\mathcal{P}_2(X)$ for the relative entropy

$$
\text{Ent}(u) = \int_X u \log u \, dm.
$$

Theorem (Ambrosio/Gigli/Savare ’11+).

For arbitrary metric measure spaces $(X, d, m)$ satisfying $CD(K, \infty)$ both approaches coincide.
Ricci Bound $\text{CD}(K, \infty)$

$\Rightarrow$

$W_2$-contraction

$\Rightarrow$

Bakry-Émery gradient estimate

$\Rightarrow$

Bochner Inequality (without $\mathcal{N}$)

$\Rightarrow$

Hess Ent$(\cdot | m) \geq K$

$W_2(P_t\mu, P_t\nu) \leq e^{-Kt} W_2(\mu, \nu)$

$|\nabla P_t u|^2 \leq e^{-2Kt} P_t |\nabla u|^2$

$\frac{1}{2} \Delta |\nabla u|^2 - \langle \nabla u, \nabla \Delta u \rangle \geq K \cdot |\nabla u|^2$

[Ambrosio, Gigli, Savare]
What are the corresponding assertions for finite $N$?

Can we deduce a $\text{CD}^*(K', N')$-condition for the transformed space $(X, d', m)$ if $(X, d, m)$ satisfies $\text{CD}^*(K, N)$ and

\[
d'(x, y) = \inf \left\{ \int_0^1 |\dot{\gamma}_t| \cdot e^{W(\gamma_t)} \, dt : \gamma : [0, 1] \to X \text{ rect.}, \gamma_0 = x, \gamma_1 = y \right\}
\]
(briefly: $d' \approx e^W d$ on infinitesimal scales)
Metric Measure Spaces with Linear Heat Flow

- Ricci Bound $\text{CD}(K, \infty)$
- $W_2$-contraction
- Bakry-Émery gradient estimate
- Bochner Inequality (without $N$)

\[ \text{Hess Ent}(\cdot | m) \geq K \]
\[ W_2(P_t \mu, P_t \nu) \leq e^{-Kt} W_2(\mu, \nu) \]
\[ |\nabla P_t u|^2 \leq e^{-2Kt} P_t |\nabla u|^2 \]
\[ \frac{1}{2} \Delta |\nabla u|^2 - \langle \nabla u, \nabla \Delta u \rangle \geq K \cdot |\nabla u|^2 \]

- What are the corresponding assertions for finite $N$?
- Can we deduce a $\text{CD}^*(K', N')$-condition for the transformed space $(X, d', m)$ if $(X, d, m)$ satisfies $\text{CD}^*(K, N)$ and

\[ d'(x, y) = \inf \left\{ \int_0^1 |\dot{\gamma}_t| \cdot e^{W(\gamma_t)} \, dt : \gamma : [0, 1] \to X \text{ rect.} , \gamma_0 = x, \gamma_1 = y \right\} \]

(briefly: $d' \approx e^W d$ on infinitesimal scales)

**Well known:** $\text{CD}^*(K, N)$ for $(X, d, m)$ implies $\text{CD}^*(K', N')$ for the weighted space $(X, d, e^V m)$ with $K' = K - \sup \left[ \text{Hess} \, V + \frac{1}{N'-N} \nabla V \otimes \nabla V \right]$
Alternative Introduction
\( \Gamma \)-Calculus of Bakry, Emery, Ledoux

**Setting**

- \( L \) linear operator defined on algebra \( \mathcal{A} \) of functions on \( X \)
  - e.g. \( L = \Delta \) Laplace-Beltrami, \( \mathcal{A} = C^\infty_c(M) \), \( X = \text{Riem.mfd.} \ M \)

**Derived quantities**

- Square field operator
  \[
  \Gamma(f, g) = \frac{1}{2} [L(fg) - fLg - gLf]
  \]

- Hessian
  \[
  H_f(g, h) = \frac{1}{2} [\Gamma(g, \Gamma(f, h)) + \Gamma(h, \Gamma(f, g)) - \Gamma(f, \Gamma(g, h))]
  \]

- \( \Gamma^2 \)-operator
  \[
  \Gamma^2(f, g) = \frac{1}{2} [L\Gamma(f, g) - \Gamma(f, Lg) - \Gamma(g, Lf)]
  \]

- e.g. \( \Gamma(f, g) = \nabla f \nabla g \), \( H_f(g, h) = \text{Hess} f(\nabla g, \nabla h) \),

\[
\Gamma^2(f, f) = \frac{1}{2} \Delta(\|\nabla f\|^2) - \nabla f \nabla \Delta f = \text{Ric}(\nabla f, \nabla f) + \|\nabla^2 f\|_{HS}^2
\]
### Setting

$L$ linear operator defined on algebra $\mathcal{A}$ of functions on $X$

e.g. $L = \Delta$ Laplace-Beltrami, $\mathcal{A} = C_c^\infty(M)$, $X = \text{Riem.mfd. } M$

### Derived quantities

- **Square field operator**
  \[ \Gamma(f, g) = \frac{1}{2} [L(fg) - fLg - gLf] \]

- **Hessian**
  \[ H_f(g, h) = \frac{1}{2} [\Gamma(g, \Gamma(f, h)) + \Gamma(h, \Gamma(f, g)) - \Gamma(f, \Gamma(g, h))] \]

- **$\Gamma_2$-operator**
  \[ \Gamma_2(f, g) = \frac{1}{2} [L\Gamma(f, g) - \Gamma(f, Lg) - \Gamma(g, Lf)] \]

e.g. $\Gamma(f, g) = \nabla f \nabla g$, $H_f(g, h) = \text{Hess}_f(\nabla g, \nabla h)$,

\[ \Gamma_2(f, f) = \frac{1}{2} \Delta(\|\nabla f\|^2) - \nabla f \nabla \Delta f = \text{Ric}(\nabla f, \nabla f) + \|\nabla^2 f\|_{HS}^2 \]

### Ricci tensor

- $R(f)(x) = \inf \{ \Gamma_2(\tilde{f})(x) : \tilde{f} \in \mathcal{A}, \Gamma(\tilde{f} - f)(x) = 0 \}$
Ricci and $N$-Ricci tensor

- $R(f)(x) = \inf \{ \Gamma_2(\tilde{f})(x) : \tilde{f} \in \mathcal{A}, \Gamma(\tilde{f} - f)(x) = 0 \}$
- $R_N(f)(x) = \inf \{ \Gamma_2(\tilde{f})(x) - \frac{1}{N}(\tilde{L}f)^2(x) : \tilde{f} \in \mathcal{A}, \Gamma(\tilde{f} - f)(x) = 0 \}$

Bakry-Emery Condition (Bochner Inequality)

$$BE(K, N) \iff \Gamma_2(f) \geq K \cdot \Gamma(f) + \frac{1}{N}(Lf)^2$$
$$\iff R_N(f) \geq K \cdot \Gamma(f)$$
### Drift transformation

\[ L'u := Lu + \Gamma(V, u) \]

- \[ R'(u) = R(u) - H_V(u, u) \]
- \[ R'_{N'}(u) \geq R_N - H_V(u, u) - \frac{1}{N' - N} \Gamma(V, u)^2 \]

**Cor.** \( \text{BE}(K,N) \) for \( L \) \( \Rightarrow \) \( \text{BE}(K',N') \) for \( L' \)

**Ex.** \( L \) generator of Dirichlet form \( \int \Gamma(u)dm \) on \( L^2(X, m) \)

\( \Rightarrow \) \( L' \) generator of Dirichlet form \( \int \Gamma(u)e^Vdm \) on \( L^2(X, e^Vm) \)

### Aim. Independent choice of weights

\[ E'(u) = \int \Gamma(u)e^{V - 2W}dm \text{ on } \mathcal{H}' = L^2(X, e^Vm) \]

\( e.g. \)

- \( W = 0 \): drift transformation
- \( V = 2W \): time change
- \( V = NW \): conformal transformation
Assume that $\mathcal{E}(u) = \int \Gamma(u) dm$ on $\mathcal{H} = L^2(X, m)$ satisfies $BE(K, N)$.

**Theorem (St. 2014)**

Then

$$\mathcal{E}'(u) = \int \Gamma(u)e^{V-2W} dm \text{ on } \mathcal{H}' = L^2(X, e^V m)$$

satisfies $BE(K', N')$.

- $W = 0$ (Drift Transformation)
  $$K' = K - \sup \left[ \text{Hess} V(\nabla f, \nabla f) + \frac{1}{N' - N} \langle \nabla V, \nabla f \rangle^2 \right] / |\nabla f|^2$$

- $V = 2W$ (Time Change)
  $$K' = \inf \left[ e^{-2W} K + \frac{1}{2} \Delta e^{-2W} - N^* |\nabla e^{-W}|^2 \right]$$

- $V = NW$ (Conformal Transformation)
  $$N' = N, \; K' = \ldots$$
What do we gain if we have \( BE(K, N) \) instead of \( BE(K, \infty) \)?

Consider evolution

\[
dX_t = \sqrt{2\alpha} dB_t - \nabla V(X_t) \, dt
\]

for \( A = \alpha \Delta - \nabla V \cdot \nabla \) on \( n \)-dimensional Riem \((M, g)\). Then

\[
BE(K, \infty) \iff \alpha \text{Ric} + \text{Hess } V \geq K
\]

\[
BE(K, N) \iff \alpha \text{Ric} + \text{Hess } V - \frac{1}{N - \alpha n} (\nabla V \otimes \nabla V) \geq K
\]

For \( \alpha \to 0 \):

\[
dX_t = -\nabla V(X_t) dt
\]

\[
\text{Hess } V - \frac{1}{N} (\nabla V \otimes \nabla V) \geq K
\]
\((K, N)\)-Convexity

**Def** \(V\) is \((K, N)\)-convex

\[\iff\text{Hess } V - \frac{1}{N} (\nabla V \otimes \nabla V) \geq K\]

\[\iff\text{Hess } U_N \leq -\frac{K}{N} \cdot U_N\quad\text{where } U_N(x) := \exp \left(-\frac{1}{N} V(x)\right)\]

\[\iff\text{ } U_N(\gamma_t) \geq \sigma_{K,N}^{(1-t)}(|\dot{\gamma}|) \cdot U_N(\gamma_0) + \sigma_{K,N}^{(t)}(|\dot{\gamma}|) \cdot U_N(\gamma_1)\]

**Example.** For \(N > 0\) and \(K > 0\)

\[V(x) = -N \log \cos \left(x \sqrt{\frac{K}{N}}\right) \geq \frac{K}{2} x^2\]

defined on \((-\sqrt{N/K\pi/2}, \sqrt{N/K\pi/2})\)
A smooth curve $x : [0, \infty) \to M$ is a solution to the gradient flow equation

$$\dot{x}_t = -\nabla V(x_t)$$

if and only if the Evolution Variation Inequality $EVI_{K,N}$ holds:

$$\frac{d}{dt} s_{K/N}^2 \left( \frac{1}{2} d(x_t, z) \right) + K \cdot s_{K/N}^2 \left( \frac{1}{2} d(x_t, z) \right) \leq \frac{N}{2} \left( 1 - \frac{U_N(z)}{U_N(x_t)} \right)$$

where $s_K(r) = \sin(\sqrt{K}r)/\sqrt{K}$, $c_K(r) = \cos(\sqrt{K}r)$
Corollary. Let \((x_t), (y_t)\) be two gradient flows of \(V\) starting from \(x_0\) resp. \(y_0\). Then for all \(s, t \geq 0:\)

\[
\frac{s^2}{K/N} \left( \frac{1}{2} d(x_t, y_s) \right) \leq e^{-K(s+t)} \cdot \frac{s^2}{K/N} \left( \frac{1}{2} d(x_0, y_0) \right) + \frac{N}{K} \left( 1 - e^{-K(s+t)} \right) \frac{(\sqrt{t} - \sqrt{s})^2}{2(s + t)}
\]

If \(K = 0:\)

\[
d^2(x_t, y_s) \leq d^2(x_0, y_0) + 4N(\sqrt{t} - \sqrt{s})^2
\]

Everything makes sense also on metric spaces!

E.g. for \(V = \text{Ent}(\cdot)\) on \(M = \mathcal{P}_2(X, d)\).
Def. A metric measure space \((X, d, m)\) satisfies \(CD^e(K, N)\)
\[\iff\] \(\text{Ent(.) is } (K,N)\)-convex on \(\mathcal{P}_2(X, d)\)
\[\iff\] \(\forall \mu_0, \mu_1 \in \mathcal{P}_2(X, d) : \exists \text{ connecting geodesic } (\mu_t)_t \text{ s.t. } \forall t \in [0, 1]:\)
\[
U_N(\mu_t) \geq \sigma_{K,N}^{(1-t)}(|\dot{\mu}|) \cdot U_N(\mu_0) + \sigma_{K,N}^{(t)}(|\dot{\mu}|) \cdot U_N(\mu_1)
\]
where
\[
U_N(\mu) = \exp \left( -\frac{1}{N} \text{Ent}(\mu) \right)
\]

Thm. For non-branching mms
\[CD^e(K, N) \iff CD^*(K, N)\]
**Thm.** The following are equivalent

- strong $CD^*(K, N)$-space and linear heat flow
- strong $CD^e(K, N)$-space and linear heat flow
- $X$ is length space and each $\mu \in \mathcal{P}_2$ is starting point of $\text{EVI}_{K, N}$-gradient flow for $\text{Ent}(.)$

**Def.** $RCD^*(K, N) = CD^*(K, N)$ and linear heat flow

**Thm.**

(i) $RCD^*(K, N)$ is preserved under convergence of mms

(ii) $RCD^*(K, N)$ is preserved under tensorization of mms

(iii) $RCD^*(K, N)$ holds globally if it holds locally
**Theorem (Erbar/Kuwada/St. 2013) “W₂-contraction”**

\[
s_{K/N}^2 \left( \frac{1}{2} W_2(P_t \mu, P_s \nu) \right) \leq e^{-K(s+t)} \cdot s_{K/N}^2 \left( \frac{1}{2} W_2(\mu, \nu) \right) \\
+ \frac{N}{K} \left( 1 - e^{-K(s+t)} \right) \frac{(\sqrt{t} - \sqrt{s})^2}{2(s+t)} .
\]

with \( s_K(r) = \sin(\sqrt{Kr})/\sqrt{K} \)

**Corollary:** With \( \tau(s, t) = 2(t + \sqrt{ts} + s)/3 \)

\[
W_2(P_t \mu, P_s \nu)^2 \leq e^{-K\tau(s,t)} W_2(\mu, \nu)^2 + 2N \frac{1 - e^{-K\tau(s,t)}}{K \tau(s, t)} (\sqrt{t} - \sqrt{s})^2 ,
\]

**Theorem (Erbar/Kuwada/St. 2013) “Bakry-Ledoux gradient estimate”**

\[
|\nabla P_t f|^2 + \frac{4Kt^2}{N(e^{2Kt} - 1)} |\Delta P_t f|^2 \leq e^{-2Kt} P_t (|\nabla f|^2)
\]
Theorem (Erbar/Kuwada/St. 2013) "Bochner Inequality"

\[ \frac{1}{2} \Delta |\nabla u|^2 - \langle \nabla u, \nabla \Delta u \rangle \geq K \cdot |\nabla u|^2 + \frac{1}{N} \cdot |\Delta u|^2 \]

- Finsler spaces: Ohta-Sturm (2012+)
- $CD^e(K, \infty)$-spaces: Ambrosio-Gigli-Savaré (2012+)
- $CD^e(K, N)$-spaces: Ambrosio-Mondino-Savaré (work in progress)

Corollary (Garofalo/Mondino 2013)

\[ \Delta (\log P_t f) \geq -\frac{N}{2t} \]

Li-Yau gradient estimate, differential Harnack inequality, Gaussian heat kernel estimates
**Thm.** For length space with linear heat flow the following are equivalent

(i) Strong $CD^*(K, N)$

(ii) Strong $CD^e(K, N)$

(iii) Existence of an $EVI_{K,N}$-gradient flow for the entropy starting from each point $\mu \in \mathcal{P}_2$

(iv) $W_2$-contraction estimate with parameters $K, N$

(v) Bakry-Ledoux gradient estimate

(vi) Bochner inequality $BE(K, N)$
Consequences of $(K,N)$-Convexity

**Thm. (N-LogSob Inequality)** Assume that $S$ is $(K, N)$-convex for some $0 < K, N < \infty$ and that $\inf S = 0$. Then

$$(\nabla S)^2 \geq KN \cdot \left[ \exp \left( \frac{2}{N} S \right) - 1 \right] \geq 2KS$$

**Thm. (N-HWI Inequality)** Assume that $S$ is $(K, N)$-convex for some $0 < K, N < \infty$ and that $\inf S = S(\bar{x}) = 0$. Then $\forall x_0$

$$|\nabla S|(x_0) \cdot d(x_0, \bar{x}) - \frac{K}{2} d^2(x_0, \bar{x}) \geq N \cdot \left[ \exp \left( \frac{1}{N} S(x_0) \right) - 1 \right] \geq S$$

**Thm. (N-Talagrand Inequality)** Assume that $S$ is $(K, N)$-convex for some $0 < K, N < \infty$ and that $\inf S = S(\bar{x}) = 0$. Then $\forall x_0$

$$S(x_0) \geq -N \log \cos \left( \sqrt{\frac{K}{N}} d(x_0, \bar{x}) \right) \geq \frac{K}{2} d^2(x_0, \bar{x})$$
Transformation of $RCD^*(K, N)$-Spaces

Given mms $(X, d, m)$, functions $V, W$ on $X$. Consider mms $(X, d', m')$ with $m' = e^V m$ and

$$d'(x, y) = \inf \left\{ \int_0^1 |\dot{\gamma}_t| \cdot e^{W(\gamma_t)} \, dt : \gamma : [0, 1] \to X \text{ rectifiable, } \gamma_0 = x, \gamma_1 = y \right\}.$$

Associated Dirichlet form: $\int |\nabla u|^2 e^{V-2W} \, dm$ on $L^2(X, e^V m)$

**Theorem (St. 2014)**

If $(X, d, m)$ satisfies $RCD^*(K, N)$ then for each $N' > N$ there exists $K'$ s.t. $(X, d', m')$ satisfies $RCD^*(K', N')$.

- $W = 0$ (Drift Transformation)
  $$K' = K - \sup \left[ \text{Hess} V(\nabla f, \nabla f) + \frac{1}{N' - N} \langle \nabla V, \nabla f \rangle^2 \right] / |\nabla f|^2$$

- $V = 2W$ (Time Change)
  $$K' = \inf \left[ e^{-2W} K + \frac{1}{2} \Delta e^{-2W} - N^* |\nabla e^{-W}|^2 \right]$$

- $V = NW$ (Conformal Transformation)
  $$N' = N, \ K' = \ldots$$
Bishop-Gromov Volume Growth Estimate

\[ \frac{s(r)}{s(R)} \geq \sin \left( \sqrt{\frac{K}{N-1}} r \right)^{N-1} \frac{\sin \left( \sqrt{\frac{K}{N-1}} R \right)^{N-1}}{\sin \left( \sqrt{\frac{K}{N-1}} R \right)} \]

Bonnet-Myers Diameter Bound

\[ \text{diam}(X) \leq \sqrt{\frac{N-1}{K}} \cdot \pi \]

Poincaré / Lichnerowicz Inequality

\[ \lambda_1 \geq \frac{N}{N-1} K \]
# Geometry of $RCD^*(K, N)$-Spaces

**Theorem (Gigli 2013) "Splitting Theorem"

If $(X, d, m)$ satisfies $RCD^*(0, N)$ and contains a line then

$$X = \mathbb{R} \times X'$$

for some $RCD^*(0, N - 1)$-space $(X', d', m')$.

**Theorem (Ketterer 2014) "Maximal Diameter Theorem"

If $(X, d, m)$ satisfies $RCD^*(N - 1, N)$ and has diameter $\pi$ then $X$ is the spherical suspension of some $RCD^*(N - 2, N - 1)$-space $(X', d', m')$.

**Theorem (Mondino/Naber 2014)

If $(X, d, m)$ satisfies $RCD^*(K, N)$ then $\exists$ integer $n \leq N$ s.t. for $m$-a.e. $x \in X$ the tangent cone at $x$ is unique and isometric to $\mathbb{R}^n$. 
The End