One-dimensional pressureless gas systems with/without viscosity

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1. The problem
   - Formulation
   - Previous results
2. Approach via scalar Conservation Laws
   - A scalar CL associated to the system
   - Entropy solution
   - Existence & uniqueness
   - Stability
3. An iterative scheme for the entropy solution
   - Discretization for the CL
   - Iterative scheme

Pressureless Gas w/ potential and viscosity

- Let $\alpha,\ \beta \in \mathbb{R},\ \lambda \in [0,\infty)$ and consider the system

$$\begin{cases} \partial_t\rho + \partial_x(\rho v) = \lambda\partial^2_{xx}\rho \\ \partial_t(\rho v) + \partial_x\big(\rho v^2\big) = \lambda\partial_x\big(v\partial_x\rho\big) + \big(\alpha\partial_x\Phi + \beta\big)\rho \quad \text{in } (0,\infty)\times\mathbb{R}. \\ \partial^2_{xx}\Phi = \rho \end{cases} \tag{P}$$

- Our concern is the initial value/Cauchy problem associated with (P), i.e. (P) along with

$$\rho(0,\cdot) = \rho^0 \quad \text{and} \quad v(0,\cdot) = v^0,\ \rho^0 - \text{a.e.} \tag{IC}$$

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Sticky particles

- Zeldovich (1970) introduced the system with $\beta = \lambda = 0$ to model the formation of large structures in the Universe.
- Let $m_1, ..., m_n$ (say, adding up to the unit) be positive masses located at $x_1^0, ..., x_n^0$ respectively, and moving initially with velocities $v_1^0, ..., v_n^0$ respectively.
- Laws of motion:
  1. Motion is rectilinear at constant velocity between collisions.
  2. When two masses collide, they stick together (additively).
  3. The velocity after collision is provided by imposing conservation of momentum through collisions.
- Let $\mu_t := \sum_{i=1}^{n(t)} m_i^{n(t)}(t)\delta_{x_i^{n(t)}(t)}$, and let v_t be defined on the support of μ_t by

  $v_t(x_i^{n(t)}) = v_i^{n(t)}$ =velocity of the mass $m_i^{n(t)}(t)$ at location $x_i^{n(t)}(t)$.
- Then (μ, v) solves (P) with $\alpha = \beta = \lambda = 0$ in the sense of distributions, with $\mu_0 = \sum_{i=1}^{n} m_i \delta_{x_i^0}$ and $v_0(x_i^0) = v_i^0$.

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Weak Solutions; early results

- We seek solutions in the sense of distributions.
- Let ρ^0 be a Borel probability and v^0 be ρ^0–measurable. The IV is in the sense

$$\lim_{t\to 0^+}\int_{\mathbb{R}}\zeta(x)\rho(t,dx)=\int_{\mathbb{R}}\zeta(x)\rho^0(dx),\ \lim_{t\to 0^+}\int_{\mathbb{R}}v(t,x)\zeta(x)\rho(t,dx)=\int_{\mathbb{R}}v^0(x)\zeta(x)\rho^0(dx)$$

  for all $\zeta\in C_c^\infty(\mathbb{R})$.
- Physically, we are looking for solutions (ρ, v) such that $\rho(t,\cdot)$ is a probability at all $t\geq 0$.
- E, Rykov & Sinai (1996) proved (variationally) existence of global solutions for $\beta=\lambda=0$ and $\alpha=0,\ -1$. Main restriction: ρ^0 is either discrete or absolutely continuous w.r.t. Lebesgue measure (more restrictions on ρ^0 and v^0).
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Flux function from initial data

- Let N^0 be the optimal right-continuous map pushing $\chi := \mathcal{L}^1|_{(0,1)}$ forward to ρ^0 and set $V^0 := v^0 \circ N^0$. Let M^0 be the c.d.f. of ρ^0.
- Let $\tilde{F} : [0, \infty) \times [0, 1] \to \mathbb{R}$ be given by

$$\tilde{F}(t, m) := \int_0^m V^0(\omega) d\omega + t \int_0^m a(\omega) d\omega = \int_0^m V^0(\omega) d\omega + t(\alpha \frac{m^2}{2} + \beta m).$$

- Then the c.d.f. $M(t, \cdot)$ of $\rho(t, \cdot)$ *formally* solves

$$\begin{cases} \partial_t M + \partial_x \left[\tilde{F}(t, M)\right] = \lambda \partial^2_{xx} M & \text{in} \quad (0, T) \times \mathbb{R}, \\ M(0, \cdot) = M^0 & \text{in} \quad \mathbb{R}. \end{cases} \qquad (CL)$$

- Conversely, $(\partial_x M, \partial_m \tilde{F}(t, M))$ *formally* solves (P) if M solves (CL).

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$$\begin{cases} \partial_t M + \partial_x\left[\tilde{F}(t,M)\right] = \lambda\partial^2_{xx}M & \text{in} \quad (0,T)\times\mathbb{R}, \\ M(0,\cdot) = M^0 & \text{in} \quad \mathbb{R}. \end{cases} \qquad (CL)$$

- Conversely, $(\partial_x M, \partial_m \tilde{F}(t,M))$ *formally* solves (P) if M solves (CL).

Flux function from initial data

- Let N^0 be the optimal right-continuous map pushing $\chi := \mathcal{L}^1|_{(0,1)}$ forward to ρ^0 and set $V^0 := v^0 \circ N^0$. Let M^0 be the c.d.f. of ρ^0.
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Entropy solution for *(CL)*

- Consider the parabolic-hyperbolic equation

$$\begin{cases} \partial_t u + \partial_x [F(t,u)] = \lambda \, u_{xx} & \text{in} \quad Q_T := (0,T) \times \mathbb{R}, \\ u(0,\cdot) = u^0 & \text{in} \quad \mathbb{R}, \end{cases} \tag{1}$$

where $\lambda \geq 0$ and the flux function has the form $F(t,z) = f(z) + tg(z)$ with continuous $f, g : \mathbb{R} \to \mathbb{R}$.

- Definition. Let $u^0 \in L^1_{loc}(\mathbb{R})$. A function $u \in L^\infty(Q_T) \cap C([0,T]; L^1_{loc}(\mathbb{R}))$ is an entropy solution of (1) if $\lambda u \in L^2(0,T; H^1_{loc}(\mathbb{R}))$ and

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- Definition. (ρ, v) is an *entropy solution* for (P)+(IC) if it is a solution in the sense of distributions and the c.d.f. M of ρ is an entropy solution for (CL) with the corresponding flux function and initial $M^0 := \mathrm{c.d.f}(\rho^0)$.
- The plan is to construct the entropy solution for (CL), then show that $M(t,\cdot)$ stays a probability c.d.f. for all times, and, finally, show that its BV derivative $\rho(t,\cdot)$ and the velocity v of the path $t \to \rho(t,\cdot)$ solve (P) in the sense of distributions.
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Assumptions and main existence & uniqueness result

- Let $p \in [1, \infty)$. We shall need the following assumptions:
  $(H1)$ *The initial distribution of mass satisfies* $\rho^0 \in \mathcal{P}_p(\mathbb{R})$;
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Theorem Nguyen & T.

If $2 \leq p < \infty$, α, $\beta \in \mathbb{R}$ and $\lambda \geq 0$, then (P)–(IC) admits a unique entropy solution (ρ, v) provided that $(H1)$ and $(H2)$ hold. Moreover, $\rho(t,\cdot) \in \mathcal{P}_p(\mathbb{R})$, $v(t,\cdot) \in L^p(\rho(t,\cdot))$ for every $t > 0$, and (i) $\|v(t,\cdot)\|_{L^p(\rho(t,\cdot))} \leq \|v^0\|_{L^p(\rho^0)} + t\|a\|_{L^p(0,1)}$ for all $t > 0$. In particular, this p–energy is nonincreasing in time if $\alpha = \beta = 0$.
(ii) Assume $\lambda > 0$. Then, $\rho(t,\cdot) \ll \mathcal{L}^1$ for $t > 0$ and its density (still denoted by ρ) satisfies

$$\rho \in L^2(0,T; L^2_{loc}(\mathbb{R})), \quad \partial_x \rho \in L^1_{loc}((0,\infty) \times \mathbb{R}) \quad \text{and} \quad \int_r^s \int_K \left| \frac{\partial_x \rho(t,x)}{\rho(t,x)} \right|^2 \rho(t,x)\, dx dt < \infty$$

for any $T > 0$, any $0 < r < s < \infty$ and any $K \Subset \mathbb{R}$.

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Moreover,

$$\|v(t,\cdot)\|_{L^p(\rho(t,\cdot))} = \|ta + V^0\|_{L^p(0,1)} \quad \text{for all} \quad t > 0.$$

In particular, the p-energy is conserved when $\alpha = \beta = 0$, i.e., $\|v(t,\cdot)\|_{L^p(\rho(t,\cdot))} = \|v^0\|_{L^p(\rho^0)}$.

(iii) $\mathcal{W}_p(\rho(t,\cdot),\rho(s,\cdot)) \leq \left(\|v^0\|_{L^p(\rho^0)} + \frac{t+s}{2}\|a\|_{L^p(0,1)}\right)|t-s|$ for all $t,s \geq 0$.

(iv) As $t \to 0^+$, $v(t,\cdot)$ converges strongly to v^0 in L^p. More generally,

$$\lim_{t\to 0^+} \int_{\mathbb{R}} h(x, v(t,x))\, \rho(t,dx) = \int_{\mathbb{R}} h(x, v^0(x))\, \rho^0(dx)$$

for all $h \in C(\mathbb{R}\times\mathbb{R})$ with at most p–growth, i.e., $|h(x,y)| \leq A + B\big(|x|^p + |y|^p\big)$ for some $A, B \geq 0$.

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Main stability result

Theorem Nguyen & T.

Assume that the conditions for existence $(H1)$ and $(H2)$ are satisfied. Let (ρ_1, v_1) and (ρ_2, v_2) be the entropy solutions for the system (P) corresponding to the initial data (ρ_1^0, v_1^0), (ρ_2^0, v_2^0), and to the triplets of constants $(\alpha_1, \beta_1, \lambda_1)$, $(\alpha_2, \beta_1, \lambda_2)$, respectively. We have:
(i) If $\lambda_1 = \lambda_2$, then, for every $t \geq 0$,

$$\mathcal{W}_p(\rho_1(t,\cdot), \rho_2(t,\cdot)) \leq \mathcal{W}_p(\rho_1^0, \rho_2^0) + t\|v_1^0 \circ N_1^0 - v_2^0 \circ N_2^0\|_{L^p(0,1)} + \frac{1}{2}t^2 \mathcal{N}(\alpha_1 - \alpha_2, \beta_1 - \beta_2; p),$$

where $\mathcal{N}(a, b; p)$ denotes the $L^p(0,1)$ norm of the function $x \mapsto ax + b$.
(ii) If $v_1^0 \circ N_1^0 \equiv v_2^0 \circ N_2^0$ Lebesgue a.e. in $(0,1)$ and $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$, then, for every $t \geq 0$,

$$\mathcal{W}_2^2(\rho_1(t,\cdot), \rho_2(t,\cdot)) \leq \mathcal{W}_2^2(\rho_1^0, \rho_2^0) + 2t\left|\sqrt{\lambda_1} - \sqrt{\lambda_2}\right|^2.$$

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(iii) In general,

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An operator on the set of c.d.f.'s

- For $h_1, h_2 > 0$ and $M \in \mathcal{M}$ (set of all c.d.f.'s), let

$$T(\lambda; h_1, h_2)M(x) := \int_{\mathbb{R}} \int_0^1 \bar{M}\Big(x - \sqrt{2\lambda h_1}\, y - h_1 f'(w) - h_2 g'(w), w\Big) G(y) dw dy,$$

where G is the standard Gaussian on $\mathbb{R}$ given by $G(y) := \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$ and $\bar{M} : \mathbb{R} \times (0,1) \to \{0,1\}$ given by $\bar{M}(x,w) := 1_{\{M \geq w\}}(x)$.

- To explain this choice, let us simplify to $\lambda = 0$ [Bolley, Brenier & Loeper (2005)]. Let $M \in \mathcal{M}$ be some fixed initial profile with generalized inverse X. Let X evolve, for $f,\ g \in W^{1,1}(0,1)$, according to

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More on this operator

- Note that $T(h_1, h_2) = T(0; h_1, h_2)$.
- So, in the case $\lambda > 0$, $T(\lambda; h_1, h_2)M$ is obtained by first mapping $M \in \mathcal{M}$ to $T(h_1, h_2)M$, and then letting $T(h_1, h_2)M$ evolve along the heat flow on the time interval $[0, h_1]$. That is, for $\lambda > 0$, we have

$$T(\lambda; h_1, h_2)M := K_{\lambda h_1} * T(h_1, h_2)M, \qquad \text{where} \quad K_h(z) := \frac{1}{\sqrt{4\pi h}} e^{-\frac{z^2}{4h}}.$$

-

$$\mathcal{W}_p\left([T(\lambda; h_1, h_2)M_1]_x, [T(\lambda; h_1, h_2)M_2]_x\right) \le \mathcal{W}_p(M_{1x}, M_{2x}).$$

-

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- Note that $T(h_1, h_2) = T(0; h_1, h_2)$.
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Iterative scheme

- Definition. Let $h > 0$ and $M \in \mathcal{M}$. For any $t \geq 0$ decomposed as $t = (N+s)h$ for a nonnegative integer $N \geq 0$ and some $0 \leq s < 1$, set

$$S_h M(t, \cdot) := (1-s)T_h^N M + sT_h^{N+1} M$$

where $T_h^0 M := M$ and $T_h^{k+1} v := T(\lambda; h, (k+0.5)h^2)T_h^k M$ for any nonnegative integer k.

Theorem Nguyen & T.

Under the assumptions $(H1)$, $(H2)$ and given any $T > 0$, S_h inherits the contraction properties of $T(\lambda; h_1, h_2)$.

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Under the assumptions $(H1)$, $(H2)$ and given any $T > 0$, as h goes to 0, the function $S_h M^0$ converges in $C([0,T]; L^1_{loc}(\mathbb{R}))$ to the unique entropy solution of (CL) with initial datum M^0.

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Thank you!