One-dimensional pressureless gas systems with/without viscosity

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- The problem
 - Formulation
 - Previous results
- Approach via scalar Conservation Laws
 - A scalar CL associated to the system
 - Entropy solution
 - Existence & uniqueness
 - Stability
- 3 An iterative scheme for the entropy solution
 - Discretization for the CL
 - Iterative scheme



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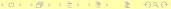
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• Let $\alpha, \beta \in \mathbb{R}, \lambda \in [0, \infty)$ and consider the system

$$\begin{cases} \partial_t \rho + \partial_x (\rho v) = \lambda \partial_{xx}^2 \rho \\ \partial_t (\rho v) + \partial_x (\rho v^2) = \lambda \partial_x (v \partial_x \rho) + (\alpha \partial_x \Phi + \beta) \rho & \text{in } (0, \infty) \times \mathbb{R}. \\ \partial_{xx}^2 \Phi = \rho \end{cases}$$
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$$\rho(0,\cdot) = \rho^0 \quad \text{and} \quad v(0,\cdot) = v^0, \ \rho^0 - \text{a.e.}$$
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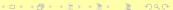
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$$\lim_{t \to 0^+} \int_{\mathbb{R}} \zeta(x) \rho(t, dx) = \int_{\mathbb{R}} \zeta(x) \rho^0(dx), \quad \lim_{t \to 0^+} \int_{\mathbb{R}} v(t, x) \zeta(x) \rho(t, dx) = \int_{\mathbb{R}} v^0(x) \zeta(x) \rho^0(dx)$$

- Physically, we are looking for solutions (ρ, v) such that $\rho(t, \cdot)$ is a probability at all $t \ge 0$.
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- Let N^0 be the optimal right-continuous map pushing $\chi := \mathcal{L}^1|_{(0,1)}$ forward to ρ^0 and set
- Let $\tilde{F}:[0,\infty)\times[0,1]\to\mathbb{R}$ be given by

$$\tilde{F}(t,m) := \int_0^m V^0(\omega) d\omega + t \int_0^m a(\omega) d\omega = \int_0^m V^0(\omega) d\omega + t (\alpha \frac{m^2}{2} + \beta m).$$

• Then the c.d.f. $M(t,\cdot)$ of $\rho(t,\cdot)$ formally solves

$$\begin{cases} \partial_t M + \partial_x \big[\tilde{F}(t, M) \big] &= \lambda \partial_{xx}^2 M & \text{in } (0, T) \times \mathbb{R}, \\ M(0, \cdot) &= M^0 & \text{in } \mathbb{R}. \end{cases}$$
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• Conversely, $(\partial_x M, \partial_m \tilde{F}(t, M))$ formally solves (P) if M solves (CL).

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• Consider the parabolic-hyperbolic equation

$$\begin{cases} \partial_t u + \partial_x \big[F(t, u) \big] &= \lambda \ u_{xx} & \text{in } Q_T := (0, T) \times \mathbb{R}, \\ u(0, \cdot) &= u^0 & \text{in } \mathbb{R}, \end{cases}$$
 (1)

• Definition. Let $u^0 \in L^1_{loc}(\mathbb{R})$. A function $u \in L^{\infty}(Q_T) \cap C([0,T];L^1_{loc}(\mathbb{R}))$ is an entropy

$$\int_{Q_T} \left\{ |u - k| \phi_t + \operatorname{sgn}(u - k) [F(t, u) - F(t, k)] \phi_x \right\} dx dt + \int_{\mathbb{R}} |u^0(x) - k| \phi(0, x) dx$$

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- The plan is to construct the entropy solution for (CL), then show that $M(t,\cdot)$ stays a probability c.d.f. for all times, and, finally, show that its BV derivative $\rho(t,\cdot)$ and the velocity v of the path $t \to \rho(t,\cdot)$ solve (P) in the sense of distributions.
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Assumptions and main existence & uniqueness result

• Let $p \in [1, \infty)$. We shall need the following assumptions: (H1) The initial distribution of mass satisfies $\rho^0 \in \mathcal{P}_p(\mathbb{R})$ (H2) The initial velocity satisfies $v^0 \in L^p(\rho^0)$.

Theorem Nguyen & T

If $2 \leq p < \infty$, α , $\beta \in \mathbb{R}$ and $\lambda \geq 0$, then (P)–(IC) admits a unique entropy solution (ρ, v) provided that (H1) and (H2) hold. Moreover, $\rho(t,\cdot) \in \mathcal{P}_p(\mathbb{R})$, $v(t,\cdot) \in L^p(\rho(t,\cdot))$ for every t>0, and $(i) \|v(t,\cdot)\|_{L^p(\rho(t,\cdot))} \leq \|v^0\|_{L^p(\rho^0)} + t\|a\|_{L^p(0,1)}$ for all t>0. In particular, this p–energy is nonincreasing in time if $\alpha=\beta=0$.

(ii) Assume $\lambda>0$. Then, $\rho(t,\cdot)\ll\mathcal{L}^1$ for t>0 and its density (still denoted by ρ) satisfies

$$ho \in L^2(0,T;L^2_{loc}(\mathbb{R})), \quad \partial_x \rho \in L^1_{loc}((0,\infty) \times \mathbb{R}) \quad \text{and} \quad \int_x^s \int_K \left| \frac{\partial_x \rho(t,x)}{\rho(t,x)} \right|^2 \rho(t,x) \, dx dt < \infty$$

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Existence & uniqueness (continued)

Theorem Nguyen & T. (continued)

Moreover,

$$||v(t,\cdot)||_{L^p(a(t,\cdot))} = ||ta+V^0||_{L^p(0,1)}$$
 for all $t>0$

In particular, the p-energy is conserved when $\alpha = \beta = 0$, i.e., $\|v(t,\cdot)\|_{L^p(\rho(t,\cdot))} = \|v^0\|_{L^p(\rho^0)}$.

$$(iii) \ \mathcal{W}_{p}(\rho(t,\cdot),\rho(s,\cdot)) \leq \left(\|v^{0}\|_{L^{p}(\rho^{0})} + \frac{t+s}{2} \|a\|_{L^{p}(0,1)} \right) |t-s| \quad \text{for all} \quad t,s \geq 0$$

(iv) As $t \to 0^+$, $v(t,\cdot)$ converges strongly to v^0 in L^p . More generally

$$\lim_{t \to 0^+} \int_{\mathbb{R}} h(x, v(t, x)) \, \rho(t, dx) = \int_{\mathbb{R}} h(x, v^0(x)) \, \rho^0(dx)$$

for all $h\in C(\mathbb{R} imes\mathbb{R})$ with at most p-growth, i.e., $|h(x,y)|\leq A+B\big(|x|^p+|y|^p\big)$ for some $A,B\geq 0$.

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Existence & uniqueness (continued)

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Main stability result

Theorem Neuven & T

Assume that the conditions for existence (H1) and (H2) are satisfied. Let (ρ_1, v_1) and (ρ_2, v_2) be the entropy solutions for the system (P) corresponding to the initial data (ρ_1^0, v_1^0) , (ρ_2^0, v_2^0) , and to the triplets of constants $(\alpha_1, \beta_1, \lambda_1)$, $(\alpha_2, \beta_1, \lambda_2)$, respectively. We have: (i) If $\lambda_1 = \lambda_2$, then, for every $t \geq 0$.

$$\mathcal{W}_{p}(\rho_{1}(t,\cdot),\rho_{2}(t,\cdot)) \leq \mathcal{W}_{p}(\rho_{1}^{0},\rho_{2}^{0}) + t \|v_{1}^{0} \circ N_{1}^{0} - v_{2}^{0} \circ N_{2}^{0}\|_{L^{p}(0,1)} + \frac{1}{2}t^{2}\mathcal{N}(\alpha_{1} - \alpha_{2},\beta_{1} - \beta_{2};p),$$

where $\mathcal{N}(a,b;p)$ denotes the $L^p(0,1)$ norm of the function $x\mapsto ax+b$.

(ii) If
$$v_1^0 \circ N_1^0 \equiv v_2^0 \circ N_2^0$$
 Lebesgue a.e. in $(0,1)$ and $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$, then, for every $t \geq 0$,

$$W_2^2(\rho_1(t,\cdot),\rho_2(t,\cdot)) \le W_2^2(\rho_1^0,\rho_2^0) + 2t|\sqrt{\lambda_1} - \sqrt{\lambda_2}|^2$$

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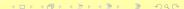
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Main stability result (continued)

Theorem Nguyen & T. (continued)

(iii) In general

$$\mathcal{W}_{2}(\rho_{1}(t,\cdot),\rho_{2}(t,\cdot)) \leq \mathcal{W}_{2}(\rho_{1}^{0},\rho_{2}^{0}) + t \|v_{1}^{0} \circ N_{1}^{0} - v_{2}^{0} \circ N_{2}^{0}\|_{L^{2}(0,1)}$$
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Main stability result (continued)

Theorem Nguyen & T. (continued)

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An operator on the set of c.d.f.'s

• For $h_1, h_2 > 0$ and $M \in \mathcal{M}$ (set of all c.d.f.'s), let

$$T(\lambda; h_1, h_2)M(x) := \int_{\mathbb{R}} \int_0^1 \bar{M}\Big(x - \sqrt{2\lambda h_1} \ y - h_1 f'(w) - h_2 g'(w), w\Big) G(y) dw dy,$$

where G is the standard Gaussian on $\mathbb R$ given by $G(y):=\frac{1}{\sqrt{2\pi}}e^{-y^2/2}$ and $\bar M:\mathbb R\times(0,1)\to\{0,1\}$ given by $\bar M(x,w):=1$ (x)

• To explain this choice, let us simplify to $\lambda=0$ [Bolley, Brenier & Loeper (2005)]. Let $M\in\mathcal{M}$ be some fixed initial profile with generalized inverse X. Let X evolve, for $f,\ g\in W^{1,1}(0,1)$, according to

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- Note that $T(h_1, h_2) = T(0; h_1, h_2)$.
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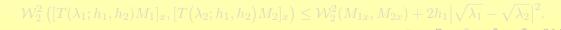
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• Definition. Let h > 0 and $M \in \mathcal{M}$. For any $t \ge 0$ decomposed as t = (N+s)h for a nonnegative integer $N \ge 0$ and some $0 \le s < 1$, set

$$S_h M(t, \cdot) := (1 - s) T_h^N M + s T_h^{N+1} M$$

where $T_h^0M:=M$ and $T_h^{k+1}v:=T(\lambda;h,(k+0.5)h^2)T_h^kM$ for any nonnegative integer k

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Under the assumptions (H1), (H2) and given any T>0, S_h inherits the contraction properties of $T(\lambda; h_1, h_2)$.

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Under the assumptions (H1), (H2) and given any T>0, as h goes to 0, the function S_hM^0 converges in $C([0,T];L^1_{loc}(\mathbb{R}))$ to the unique entropy solution of (CL) with initial datum M^0 .

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Thank you!