Equilibrium in risk-sharing games

Michail Anthropelos (University of Piraeus)
joint with
Constantinos Kardaras (LSE)

Mathematical Finance: Arbitrage and Portfolio Optimization
Banff
May, 2014
**Motivation**

- Financial agents sharing their risky position by designing new financial contracts in a mutually beneficial way.
- Such risk sharing involves only a small number of agents. Each agent can influence the equilibrium sharing; → not a cooperative equilibrium.
- Agents’ strategic behaviour in risk sharing should be introduced.

We ask:

- How much risk should an agent share? (Best response problem)
- How and at which point the market equilibrate? (Nash equilibrium)
- Do certain agents benefit from the game? (Equilibria comparison)

**(Very) short list of related literature**

- On optimal risk sharing: Seminal works of Borch ['62, '68] and Wilson ['68]. See also Duffie & Rahi ['95], Barrieu & El Karoui ['04, '05], Jouini, Schachermayer & Touzi ['08] etc.
- Non-cooperative risk sharing games: Horst & Moreno-Bromberg ['08, '12] (adverse selection), Vayanos ['99], Carvajal et al. ['11], Rostek & Weretka ['12] etc.
Motivation

- Financial agents sharing their risky position by designing new financial contracts in a mutually beneficial way.
- Such risk sharing involves only a small number of agents. Each agent can influence the equilibrium sharing; → not a cooperative equilibrium.
- Agents’ strategic behaviour in risk sharing should be introduced.

We ask:

- How much risk should an agent share? (Best response problem)
- How and at which point the market equilibrate? (Nash equilibrium)
- Do certain agents benefit from the game? (Equilibria comparison)

(Very) short list of related literature

- On optimal risk sharing: Seminal works of Borch ['62, '68] and Wilson ['68]. See also Duffie & Rahi ['95], Barrieu & El Karoui ['04, '05], Jouini, Schachermayer & Touzi ['08] etc.
- Non-cooperative risk sharing games: Horst & Moreno-Bromberg ['08, '12] (adverse selection), Vayanos ['99], Carvajal et al. ['11], Rostek & Weretka ['12] etc.
Motivation

- Financial agents sharing their risky position by designing new financial contracts in a mutually beneficial way.
- Such risk sharing involves only a small number of agents. Each agent can influence the equilibrium sharing; → not a cooperative equilibrium.
- Agents’ strategic behaviour in risk sharing should be introduced.

We ask:

✓ How much risk should an agent share? (Best response problem)
✓ How and at which point the market equilibrate? (Nash equilibrium)
✓ Do certain agents benefit from the game? (Equilibria comparison)

(Very) short list of related literature

- On optimal risk sharing: Seminal works of Borch ['62, '68] and Wilson ['68]. See also Duffie & Rahi ['95], Barrieu & El Karoui ['04, '05], Jouini, Schachermayer & Touzi ['08] etc.
- Non-cooperative risk sharing games: Horst & Moreno-Bromberg ['08, '12] (adverse selection), Vyanos ['99], Carvajal et al. ['11], Rostek & Weretka ['12].
Motivation

- Financial agents sharing their risky position by designing new financial contracts in a mutually beneficial way.
- Such risk sharing involves only a small number of agents. Each agent can influence the equilibrium sharing; → not a cooperative equilibrium.
- Agents’ strategic behaviour in risk sharing should be introduced.

We ask:

✓ **How much risk should an agent share?** (Best response problem)
✓ **How and at which point the market equilibrate?** (Nash equilibrium)
✓ **Do certain agents benefit from the game?** (Equilibria comparison)

(Very) short list of related literature

- On optimal risk sharing: Seminal works of Borch ['62, '68] and Wilson ['68]. See also Duffie & Rahi ['95], Barrieu & El Karoui ['04, '05], Jouini, Schachermayer & Touzi ['08] etc.
- Non-cooperative risk sharing games: Horst & Moreno-Bromberg ['08, '12] (adverse selection), Vayanos ['99], Carvajal et al. ['11], Rostek & Weretka ['12] etc.
Motivation

- Financial agents sharing their risky position by designing new financial contracts in a mutually beneficial way.
- Such risk sharing involves only a small number of agents. Each agent can influence the equilibrium sharing; → not a cooperative equilibrium.
- Agents’ strategic behaviour in risk sharing should be introduced.

We ask:

✓ How much risk should an agent share? (Best response problem)
✓ How and at which point the market equilibrate? (Nash equilibrium)
✓ Do certain agents benefit from the game? (Equilibria comparison)

(Very) short list of related literature

- On optimal risk sharing: Seminal works of Borch ['62, '68] and Wilson ['68]. See also Duffie & Rahi ['95], Barrieu & El Karoui ['04, '05], Jouini, Schachermayer & Touzi ['08] etc.
- Non-cooperative risk sharing games: Horst & Moreno-Bromberg ['08, '12] (adverse selection), Vayanos ['99], Carvajal et al. ['11], Rostek & Weretka ['12]
Motivation

- Financial agents sharing their risky position by designing new financial contracts in a mutually beneficial way.
- Such risk sharing involves only a small number of agents. Each agent can influence the equilibrium sharing; → not a cooperative equilibrium.
- Agents’ strategic behaviour in risk sharing should be introduced.

We ask:

✓ How much risk should an agent share? (Best response problem)
✓ How and at which point the market equilibrate? (Nash equilibrium)
✓ Do certain agents benefit from the game? (Equilibria comparison)

(Very) short list of related literature

- On optimal risk sharing: Seminal works of Borch ['62, '68] and Wilson ['68]. See also Duffie & Rahi ['95], Barrieu & El Karoui ['04, '05], Jouini, Schachermayer & Touzi ['08] etc.
Motivation

- Financial agents sharing their risky position by designing new financial contracts in a mutually beneficial way.
- Such risk sharing involves only a small number of agents. Each agent can influence the equilibrium sharing; → not a cooperative equilibrium.
- Agents’ strategic behaviour in risk sharing should be introduced.

We ask:

✓ How much risk should an agent share? (Best response problem)
✓ How and at which point the market equilibrate? (Nash equilibrium)
✓ Do certain agents benefit from the game? (Equilibria comparison)

(Very) short list of related literature

- On optimal risk sharing: Seminal works of Borch ['62, '68] and Wilson ['68]. See also Duffie & Rahi ['95], Barrieu & El Karoui ['04, ’05], Jouini, Schachermayer & Touzi ['08] etc.
Outline

1. Risk sharing and Arrow-Debreu equilibrium
2. Agent’s best endowment response
3. Nash equilibria in risk sharing
4. Extreme risk tolerance
5. Conclusive remarks & open questions
Outline

1. Risk sharing and Arrow-Debreu equilibrium
2. Agent's best endowment response
3. Nash equilibria in risk sharing
4. Extreme risk tolerance
5. Conclusive remarks & open questions
Agents and preferences

Static probability model

- $\mathbb{L}^0 \equiv \mathbb{L}^0(\Omega, \mathcal{F}, \mathbb{P})$: discounted future financial positions.
- $I = \{0, \ldots, n\}$: index set of $n + 1$ economic agents.

Preferences

- Agents’ risk preferences modelled via monetary utility functionals:
  
  $$\mathbb{L}^0 \ni X \mapsto U_i(X) := -\delta_i \log \left( \mathbb{E} \left[ \exp \left( -\frac{X}{\delta_i} \right) \right] \right) \in [-\infty, \infty).$$

- Define the aggregate risk tolerance
  
  $$\delta := \sum_{i \in I} \delta_i,$$

  as well as
  
  $$\lambda_i := \frac{\delta_i}{\delta}, \quad \delta_{-i} := \delta - \delta_i, \quad \forall i \in I.$$
Agents and preferences

Static probability model

- \( \mathbb{L}^0 \equiv \mathbb{L}^0(\Omega, \mathcal{F}, \mathbb{P}) \): discounted future financial positions.
- \( I = \{0, \ldots, n\} \): index set of \( n + 1 \) economic agents.

Preferences

- Agents’ risk preferences modelled via monetary utility functionals:
  \[
  \mathbb{L}^0 \ni X \mapsto U_i(X) := -\delta_i \log \left( \mathbb{E} \left[ \exp \left( -\frac{X}{\delta_i} \right) \right] \right) \in [-\infty, \infty].
  \]

  - Define the aggregate risk tolerance
    \[
    \delta := \sum_{i \in I} \delta_i,
    \]
    as well as
    \[
    \lambda_i := \frac{\delta_i}{\delta}, \quad \delta_{-i} := \delta - \delta_i, \quad \forall i \in I.
    \]
Agents and preferences

Static probability model

- $\mathbb{L}^0 \equiv \mathbb{L}^0(\Omega, \mathcal{F}, \mathbb{P})$: discounted future financial positions.
- $I = \{0, \ldots, n\}$: index set of $n + 1$ economic agents.

Preferences

- Agents’ risk preferences modelled via monetary utility functionals:

  $\mathbb{L}^0 \ni X \mapsto U_i(X) := -\delta_i \log \left( \mathbb{E} \left[ \exp \left( -\frac{X}{\delta_i} \right) \right] \right) \in [-\infty, \infty).$

- Define the aggregate risk tolerance

  $$\delta := \sum_{i \in I} \delta_i,$$

  as well as

  $$\lambda_i := \frac{\delta_i}{\delta}, \quad \delta_{-i} := \delta - \delta_i, \quad \forall i \in I.$$
Endowments and Contracts

Endowments

- $E_i \in \mathbb{I}^0$: random endowment (risky position) of agent $i \in I$.
- Aggregate endowment:

$$E := \sum_{i \in I} E_i.$$ 

Standing assumption enforced throughout: $(E_i)_{i \in I} \in \mathcal{E}$; in effect,

$$U_i(E_i) > -\infty, \quad \forall i \in I.$$

Sharing via contracts

$$C := \left\{ (C_i)_{i \in I} \in (\mathbb{I}^0)^I \mid \sum_{i \in I} C_i = 0 \right\}.$$ 

→ After sharing, position of agent $i \in I$ is $E_i + C_i$. 
Endowments and Contracts

Endowments

- $E_i \in \mathbb{L}^0$: random endowment (risky position) of agent $i \in I$.
- Aggregate endowment:
  \[
  E := \sum_{i \in I} E_i. 
  \]
- Standing assumption enforced throughout: $(E_i)_{i \in I} \in \mathcal{E}$; in effect,
  \[
  \mathbb{U}_i(E_i) > -\infty, \quad \forall i \in I. 
  \]

Sharing via contracts

- $C := \{ (C_i)_{i \in I} \in (\mathbb{L}^0)^I \mid \sum_{i \in I} C_i = 0 \}$.
- After sharing, position of agent $i \in I$ is $E_i + C_i$. 

Endowments and Contracts

Endowments

- $E_i \in \mathbb{L}^0$: random endowment (risky position) of agent $i \in I$.
- Aggregate endowment:
  
  \[ E := \sum_{i \in I} E_i. \]

- **Standing assumption** enforced throughout: $(E_i)_{i \in I} \in \mathcal{E}$; in effect,
  \[ \mathbb{U}_i(E_i) > -\infty, \quad \forall i \in I. \]

Sharing via contracts

\[ C := \left\{ (C_i)_{i \in I} \in (\mathbb{L}^0)^I \mid \sum_{i \in I} C_i = 0 \right\}. \]

→ After sharing, position of agent $i \in I$ is $E_i + C_i$. 
Complete market equilibrium

Arrow-Debreu equilibrium

Valuation probability $Q^*$ (equivalent to $P$) and contracts $(C_i^*)_{i \in I} \in C$ such that:

1. $\mathbb{E}_{Q^*}[C_i^*] = 0, \forall i \in I$.
2. $U_i(E_i + C_i) \leq U_i(E_i + C_i^*), \forall i \in I$ and $C_i \in \mathbb{L}^0$ with $\mathbb{E}_{Q^*}[C_i] \leq 0$.

Theorem (Borch '62)

A unique Arrow-Debreu equilibrium exists; in fact, $dQ^*/dP \propto \exp(-E/\delta)$ and

$$C_i^* := \lambda_i E - E_i - \mathbb{E}_{Q^*}[\lambda_i E - E_i], \quad \forall i \in I.$$ 

Aggregate monetary utility in Arrow-Debreu equilibrium

$(C_i^*)_{i \in I}$ is a maximiser of $C \ni (C_i)_{i \in I} \mapsto \sum_{i \in I} U_i(E_i + C_i)$; furthermore,

$$\sum_{i \in I} U_i(E_i + C_i^*) = -\delta \log \mathbb{E} [\exp (-E/\delta)] \geq \sum_{i \in I} U_i(E_i).$$

$\geq$ above is $=$ $\iff C_i^* = 0, \forall i \in I.$
Complete market equilibrium

Arrow-Debreu equilibrium

Valuation probability $Q^*$ (equivalent to $P$) and contracts $(C^*_i)_{i \in I} \in \mathcal{C}$ such that:

- $\mathbb{E}_{Q^*}[C^*_i] = 0, \forall i \in I$.
- $U_i(E_i + C_i) \leq U_i(E_i + C_i^*), \forall i \in I$ and $C_i \in \mathbb{L}^0$ with $\mathbb{E}_{Q^*}[C_i] \leq 0$.

Theorem (Borch '62)

A unique Arrow-Debreu equilibrium exists; in fact, $\frac{dQ^*}{dP} \propto \exp(-E/\delta)$ and

$$C^*_i := \lambda_i E - E_i - \mathbb{E}_{Q^*}[\lambda_i E - E_i], \quad \forall i \in I.$$ 

Aggregate monetary utility in Arrow-Debreu equilibrium

$(C^*_i)_{i \in I}$ is a maximiser of $\mathcal{C} \ni (C_i)_{i \in I} \mapsto \sum_{i \in I} U_i(E_i + C_i)$; furthermore,

$$\sum_{i \in I} U_i(E_i + C^*_i) = -\delta \log \mathbb{E}[\exp(-E/\delta)] \geq \sum_{i \in I} U_i(E_i).$$

$\rightarrow \geq$ above is $=$ $\iff C^*_i = 0, \forall i \in I$. 

M. Anthropelos (Un. Piraeus)
Complete market equilibrium

Arrow-Debreu equilibrium

Valuation probability $Q^*$ (equivalent to $P$) and contracts $(C^*_i)_{i \in I} \in C$ such that:

- $\mathbb{E}_{Q^*}[C^*_i] = 0$, $\forall i \in I$.
- $U_i(E_i + C_i) \leq U_i(E_i + C^*_i)$, $\forall i \in I$ and $C_i \in \mathbb{L}^0$ with $\mathbb{E}_{Q^*}[C_i] \leq 0$.

Theorem (Borch ’62)

A unique Arrow-Debreu equilibrium exists; in fact, $dQ^*/dP \propto \exp(-E/\delta)$ and

$$C^*_i := \lambda_i E - E_i - \mathbb{E}_{Q^*}[\lambda_i E - E_i], \quad \forall i \in I.$$ 

Aggregate monetary utility in Arrow-Debreu equilibrium

$(C^*_i)_{i \in I}$ is a maximiser of $C \ni (C_i)_{i \in I} \mapsto \sum_{i \in I} U_i(E_i + C_i)$; furthermore,

$$\sum_{i \in I} U_i(E_i + C^*_i) = -\delta \log \mathbb{E}[\exp(-E/\delta)] \geq \sum_{i \in I} U_i(E_i).$$

$\rightarrow$ “$\geq$” above is “$=$” $\iff C^*_i = 0$, $\forall i \in I$. 
Outline

1. Risk sharing and Arrow-Debreu equilibrium
2. Agent’s best endowment response
3. Nash equilibria in risk sharing
4. Extreme risk tolerance
5. Conclusive remarks & open questions
Reported endowments

Agents may have motive to report different endowments than their actual ones.

What if instead of \((E_i)_{i \in I} \in \mathcal{E}\), agents choose to report \((F_i)_{i \in I} \in \mathcal{E}\)?

- With \(F := \sum_{i \in I} F_i\), the valuation measure \(Q^F\) is such that
  \[
  \frac{dQ^F}{dP} \propto \exp\left(-\frac{F}{\delta}\right).
  \]
- Leads to risk-sharing with contracts

\[
C_i = \lambda_i F - F_i - \mathbb{E}_{Q^F}[\lambda_i F - F_i]
= \lambda_i F_{-i} - \lambda_{-i} F_i - \mathbb{E}_{Q^{F_{-i} + F_i}}[\lambda_i F_{-i} - \lambda_{-i} F_i], \quad \forall i \in I,
\]

\((\star)\)

Stage 1: Agents agree on the sharing rules of the reported endowments.

Revealed endowments via valuation measure and contracts

Given \(Q\) and \((C_i)_{i \in I} \in \mathcal{C}\) such that \(\mathbb{E}_Q[C_i] = 0, \forall i \in I\),
\(\exists (F_i)_{i \in I}\) (unique up to cash translation) such that

\[
Q = Q^F \quad \text{and} \quad (C_i)_{i \in I} \quad \text{are given by } (\star).
\]
Reported endowments

Agents may have motive to report different endowments than their actual ones.

What if instead of \((E_i)_{i \in I} \in \mathcal{E}\), agents choose to report \((F_i)_{i \in I} \in \mathcal{E}\)?

- With \(F := \sum_{i \in I} F_i\), the valuation measure \(Q^F\) is such that
  \[
  \frac{dQ^F}{dP} \propto \exp\left(-\frac{F}{\delta}\right).
  \]
- Leads to risk-sharing with contracts

  \[
  C_i = \lambda_i F - F_i - \mathbb{E}_{Q^F} [\lambda_i F - F_i]
  = \lambda_i F_{-i} - \lambda_{-i} F_i - \mathbb{E}_{Q^F_{-i} + F_i} [\lambda_i F_{-i} - \lambda_{-i} F_i], \quad \forall i \in I,
  \]

Stage 1: Agents agree on the sharing rules of the reported endowments.

Revealed endowments via valuation measure and contracts

Given \(Q\) and \((C_i)_{i \in I} \in \mathcal{C}\) such that \(\mathbb{E}_Q [C_i] = 0, \forall i \in I\)

\(\exists (F_i)_{i \in I}\) (unique up to cash translation) such that

\[
Q = Q^F \quad \text{and} \quad (C_i)_{i \in I} \quad \text{are given by} \ (\star).
\]
Reported endowments

Agents may have motive to report different endowments than their actual ones.

What if instead of \((E_i)_{i \in I} \in \mathcal{E}\), agents choose to report \((F_i)_{i \in I} \in \mathcal{E}\)?

- With \(F := \sum_{i \in I} F_i\), the valuation measure \(Q^F\) is such that
  \[
  \frac{dQ^F}{dP} \propto \exp\left(-\frac{F}{\delta}\right).
  \]
- Leads to risk-sharing with contracts

\[
C_i = \lambda_i F - F_i - \mathbb{E}_{Q^F}\left[\lambda_i F - F_i\right]
= \lambda_i F_{-i} - \lambda_{-i} F_i - \mathbb{E}_{Q^{F_{-i}+F_i}}\left[\lambda_i F_{-i} - \lambda_{-i} F_i\right], \quad \forall i \in I,
\]

(\star)

Stage 1: Agents agree on the sharing rules of the reported endowments.

 Revealed endowments via valuation measure and contracts

Given \(Q\) and \((C_i)_{i \in I} \in \mathcal{C}\) such that \(\mathbb{E}_Q[C_i] = 0, \forall i \in I\)

\(\exists (F_i)_{i \in I}\) (unique up to cash translation) such that

\[
Q = Q^F \quad \text{and} \quad (C_i)_{i \in I} \quad \text{are given by (\star)}.
\]
Reported endowments

Agents may have motive to report different endowments than their actual ones.

What if instead of \((E_i)_{i \in I} \in \mathcal{E}\), agents choose to report \((F_i)_{i \in I} \in \mathcal{E}\)?

- With \(F := \sum_{i \in I} F_i\), the valuation measure \(Q^F\) is such that
  \[ \frac{dQ^F}{dP} \propto \exp\left(-\frac{F}{\delta}\right). \]
- Leads to risk-sharing with contracts

\[
C_i = \lambda_i F - F_i - \mathbb{E}_{Q^F} [\lambda_i F - F_i] \\
= \lambda_i F_{-i} - \lambda_{-i} F_i - \mathbb{E}_{Q^F_{-i} + F_i} [\lambda_i F_{-i} - \lambda_{-i} F_i], \quad \forall i \in I, \quad (\star)
\]

Stage 1: Agents agree on the sharing rules of the reported endowments.

Revealed endowments via valuation measure and contracts

Given \(Q\) and \((C_i)_{i \in I} \in \mathcal{C}\) such that \(\mathbb{E}_Q [C_i] = 0, \forall i \in I\)

\[\exists (F_i)_{i \in I} \text{ (unique up to cash translation)} \text{ such that} \]

\[Q = Q^F \quad \text{and} \quad (C_i)_{i \in I} \quad \text{are given by (\star)}.\]
Best endowment response: the problem
Consider the position of agent \( i \in I \). Given

- the agreed mechanism that produces the optimal sharing contracts; and
- the endowment \( F_{-i} \) reported by the rest \( n \) agents in \( I \setminus \{i\} \),

a natural question is:

Which random quantity should agent \( i \in I \) report as actual endowment?

Response function

Let \( F_{-i} \) given. The **response function** of agent \( i \in I \) is

\[
V_i(F_i; F_{-i}) := U_i \left( E_i + \lambda_i F_{-i} - \lambda_i F_i - \mathbb{E}_{Q^{F_{-i} + F_i}} [\lambda_i F_{-i} - \lambda_i F_i] \right).
\]

- \( V_i(F_i + c; F_{-i}) = V_i(F_i; F_{-i}) \) holds for all \( c \in \mathbb{R} \).
- \( V_i(\cdot; F_{-i}) \) is *not* concave in general.

Best response

For given \( F_{-i} \), we seek \( F_i^r \) such that

\[
V_i(F_i^r; F_{-i}) = \sup_{F_i} V_i(F_i; F_{-i}).
\]
Best endowment response: the problem

Consider the position of agent $i \in I$. Given

- the agreed mechanism that produces the optimal sharing contracts; and
- the endowment $F_{\sim i}$ reported by the rest $n$ agents in $I \setminus \{i\}$,

a natural question is:

Which random quantity should agent $i \in I$ report as actual endowment?

Response function

Let $F_{\sim i}$ given. The response function of agent $i \in I$ is

$$V_i(F_i; F_{\sim i}) := U_i \left( E_i + \lambda_i F_{\sim i} - \lambda_{\sim i} F_i - \mathbb{E}_{Q^{F_{\sim i} + F_i}} [\lambda_i F_{\sim i} - \lambda_{\sim i} F_i] \right).$$

- $V_i(F_i + c; F_{\sim i}) = V_i(F_i; F_{\sim i})$ holds for all $c \in \mathbb{R}$.
- $V_i(\cdot; F_{\sim i})$ is not concave in general.

Best response

For given $F_{\sim i}$, we seek $F^*_i$ such that

$$V_i(F^*_i; F_{\sim i}) = \sup_{F_i} V_i(F_i; F_{\sim i}).$$
Best endowment response: the problem

Consider the position of agent $i \in I$. Given

- the agreed mechanism that produces the optimal sharing contracts; and
- the endowment $F_{-i}$ reported by the rest $n$ agents in $I \setminus \{i\}$,

a natural question is:

Which random quantity should agent $i \in I$ report as actual endowment?

Response function

Let $F_{-i}$ given. The response function of agent $i \in I$ is

$$
\mathbb{V}_i(F_i; F_{-i}) := \mathbb{U}_i \left( E_i + \lambda_i F_{-i} - \lambda^{-i}_i F_i - \mathbb{E}_{Q^{F_{-i}+F_i}} [\lambda_i F_{-i} - \lambda^{-i}_i F_i] \right).
$$

- $\mathbb{V}_i(F_i + c; F_{-i}) = \mathbb{V}_i(F_i; F_{-i})$ holds for all $c \in \mathbb{R}$.
- $\mathbb{V}_i(\cdot; F_{-i})$ is not concave in general.

Best response

For given $F_{-i}$, we seek $F^r_i$ such that

$$
\mathbb{V}_i(F^r_i; F_{-i}) = \sup_{F_i} \mathbb{V}_i(F_i; F_{-i}).
$$
Proposition (Necessary and sufficient conditions for optimality)

Let $i \in I$, $F_{-i}$ and $F_i^r$ given. The following are equivalent:

1. $\nabla_i(F_i^r; F_{-i}) = \sup_{F_i} \nabla_i(F_i; F_{-i})$.

2. $C_i^r := \lambda_i F_{-i} - \lambda_{-i} F_i^r - \mathbb{E}_{Q^{F_{-i} + F_i^r}}[\lambda_i F_{-i} - \lambda_{-i} F_i^r]$ is such that

$$\delta \frac{C_i^r}{\delta_{-i}} + \delta_i \log \left(1 + \frac{C_i^r}{\delta_{-i}}\right) = z_i^r - E_i + \delta_i \frac{F_{-i}}{\delta_{-i}},$$

(note the a-priori necessary bound $C_i^r > -\delta_{-i}$) and $z_i^r \in \mathbb{R}$ is such that

$$z_i^r = U_i(E_i + C_i^r) - U_i \left(\frac{\delta_i}{\delta_{-i}} (F_{-i} - C_i^r)\right).$$

(1) $\Rightarrow$ (2): 1st-order conditions. $\nabla_i(\cdot; F_{-i})$ is not concave: (2) $\Rightarrow$ (1) is tricky.

Theorem

There exists unique (up to constants) $F_i^r$ s.t. $\nabla_i(F_i^r; F_{-i}) = \sup_{F_i} \nabla_i(F_i; F_{-i})$. 
Best endowment response: results

Proposition (Necessary and sufficient conditions for optimality)

Let \( i \in I \), \( F_{-i} \) and \( F^r_i \) given. The following are equivalent:

1. \( \nabla_i (F^r_i; F_{-i}) = \sup_{F_i} \nabla_i (F_i; F_{-i}) \).

2. \( C^r_i := \lambda_i F_{-i} - \lambda_{-i} F^r_i - \mathbb{E}_{Q^{F_{-i}+F^r_i}} [\lambda_i F_{-i} - \lambda_{-i} F^r_i] \) is such that

\[
\delta \frac{C^r_i}{\delta_{-i}} + \delta_{-i} \log \left( 1 + \frac{C^r_i}{\delta_{-i}} \right) = z^r_i - E_i + \delta_i \frac{F_{-i}}{\delta_{-i}},
\]

(note the a-priori necessary bound \( C^r_i > -\delta_{-i} \)) and \( z^r_i \in \mathbb{R} \) is such that

\[
z^r_i = \mathbb{U}_i (E_i + C^r_i) - \mathbb{U}_i \left( \frac{\delta_{-i}}{\delta_i} (F_{-i} - C^r_i) \right).
\]

\(1) \Rightarrow (2)\): 1st-order conditions. \( \nabla_i (\cdot; F_{-i}) \) is not concave: \( (2) \Rightarrow (1) \) is tricky.

Theorem

There exists unique (up to constants) \( F^r_i \) s.t. \( \nabla_i (F^r_i; F_{-i}) = \sup_{F_i} \nabla_i (F_i; F_{-i}) \).
Proposition (Necessary and sufficient conditions for optimality)

Let \( i \in I, F_{-i} \) and \( F_i^r \) given. The following are equivalent:

1. \( \nabla_i (F_i^r; F_{-i}) = \sup_{F_i} \nabla_i (F_i; F_{-i}) \).

2. \( C_i^r := \lambda_i F_{-i} - \lambda_{-i} F_i^r - \mathbb{E}_{Q^{F_{-i} + F_i^r}} [\lambda_i F_{-i} - \lambda_{-i} F_i^r] \) is such that

\[
\delta \frac{C_i^r}{\delta_{-i}} + \delta_i \log \left( 1 + \frac{C_i^r}{\delta_{-i}} \right) = z_i^r - E_i + \delta_i \frac{F_{-i}}{\delta_{-i}},
\]

(note the \textit{a-priori} necessary bound \( C_i^r > -\delta_{-i} \)) and \( z_i^r \in \mathbb{R} \) is such that

\[
z_i^r = \mathbb{U}_i (E_i + C_i^r) - \mathbb{U}_i \left( \frac{\delta_i}{\delta_{-i}} (F_{-i} - C_i^r) \right).
\]

(1) \( \Rightarrow \) (2): 1st-order conditions. \( \nabla_i (\cdot; F_{-i}) \) is not concave: (2) \( \Rightarrow \) (1) is tricky.

Theorem

There exists unique (up to constants) \( F_i^r \) s.t. \( \nabla_i (F_i^r; F_{-i}) = \sup_{F_i} \nabla_i (F_i; F_{-i}) \).
Outline

1. Risk sharing and Arrow-Debreu equilibrium
2. Agent's best endowment response
3. Nash equilibria in risk sharing
4. Extreme risk tolerance
5. Conclusive remarks & open questions
Nash Equilibrium

Stage 2

- All agents have same strategic behaviour.
- Given the agreed risk sharing rules (stage 1), agents negotiate the contracts they are going to trade and the valuation measure.

Definition

A valuation measure $Q^\diamond$ and a collection of contracts $(C_i^\diamond)_{i \in I} \in \mathcal{C}$ will be called a game (Nash) equilibrium if

$$\forall_i (F_i^\diamond; F_{-i}^\diamond) = \sup_{F_i} \forall_i (F_i; F_{-i}^\diamond), \quad \forall i \in I,$$

where $(F_i^\diamond)_{i \in I}$ are the corresponding revealed endowments, given implicitly by

$$\frac{dQ^\diamond}{dP} \propto \exp \left(-F^\diamond / \delta \right)$$

and

$$C_i^\diamond = \lambda_i F^\diamond - F_i^\diamond - E_{Q^\diamond} [\lambda_i F^\diamond - F_i^\diamond].$$
Nash Equilibrium

Stage 2

- All agents have the same strategic behavior.
- Given the agreed risk sharing rules (stage 1), agents negotiate the contracts they are going to trade and the valuation measure.

Definition

A valuation measure $Q^\diamond$ and a collection of contracts $(C^i_i)_{i \in I} \in C$ will be called a game (Nash) equilibrium if

$$\forall i \in I, \left( F^i_i; F^i_{-i} \right) = \sup_{F_i} \forall i \in I, \\
\left( F^i_i; F^i_{-i} \right),$$

where $(F^i_i)_{i \in I}$ are the corresponding revealed endowments, given implicitly by

$$\frac{dQ^\diamond}{dP} \propto \exp \left( -\frac{F^\diamond}{\delta} \right)$$

and

$$C^i_i = \lambda_i F^\diamond - F^i_i - \mathbb{E}_{Q^\diamond} [\lambda_i F^\diamond - F^i_i].$$
Necessary and sufficient conditions for Nash equilibrium

Theorem

For given $Q^\diamondsuit$ and $(C_i^\diamondsuit)_{i \in I} \in \mathcal{C}$, the following conditions are equivalent:

1. $(Q^\diamondsuit, (C_i^\diamondsuit)_{i \in I})$ is a Nash equilibrium.

2. $C_i^\diamondsuit > -\delta_i$, and there exists $z^\diamondsuit \equiv (z_i^\diamondsuit)_{i \in I} \in \mathbb{R}^l$ with $\sum_{i \in I} z_i^\diamondsuit = 0$ such that

$$C_i^\diamondsuit + \delta_i \log \left(1 + \frac{C_i^\diamondsuit}{\delta_i} \right) = z_i^\diamondsuit + C_i^\ast + \frac{\delta_i}{\delta} \sum_{j \in I} \left(1 + \delta_j \frac{C_j^\diamondsuit}{\delta_j} \right), \quad \forall i \in I. \quad (1)$$

3. $Q^\diamondsuit$ is such that

$$\frac{dQ^\diamondsuit}{dQ^\ast} \propto \prod_{j \in I} \left(1 + \frac{C_j^\diamondsuit}{\delta_j} \right)^{\delta_j/\delta}. \quad (2)$$

4. $E_{Q^\diamondsuit} [C_i^\diamondsuit] = 0, \quad \forall i \in I.$
Existence (and uniqueness) of Nash equilibria?

In search of equilibrium

Parametrise candidate optimal contracts in

$$\Delta^I := \{(z_i)_{i \in I} \in \mathbb{R}^I \mid \sum_{i \in I} z_i = 0\} \equiv \mathbb{R}^n \quad (\text{where } n = \#I - 1).$$

- For all $$z \in \Delta^I$$, there exists a unique $$(C_i(z))_{i \in I} \in \mathcal{C}$$ satisfying equations (1).
- **Aim:** find $$z \in \Delta^I$$ such that $$E_Q(z) [C_i(z)] = 0$$ holds for all $$i \in I$$.

**Theorem 1**

In a Nash equilibrium, $$E_Q(z^\diamond) [C_i(z^\diamond)] = 0$$ holds $$\forall i \in I$$.

**Theorem 2**

Let $$z^\diamond \in \Delta^I$$ be such that $$E_Q(z^\diamond) [C_i(z^\diamond)] = 0$$ holds $$\forall i \in I$$. Then, $$(Q^\diamond, (C_i^\diamond)_{i \in I})$$ defined by (1) and (2) for $$z = z^\diamond$$ is a Nash equilibrium.

If $$I = \{0, 1\}$$, there exists a unique $$z^\diamond \in \Delta^I = \mathbb{R}$$ with $$E_Q(z^\diamond) [C_i(z^\diamond)] = 0$$, $$\forall i \in I$$. 

M. Anthropelos (Un. Piraeus)  
Equilibrium in Risk Sharing Games  
Banff 2014  
15 / 24
Existence (and uniqueness) of Nash equilibria?

In search of equilibrium

Parametrise candidate optimal contracts in

\[ \Delta^I := \{(z_i)_{i \in I} \in \mathbb{R}^I \mid \sum_{i \in I} z_i = 0 \} \equiv \mathbb{R}^n \quad (\text{where } n = \#I - 1). \]

- For all \( z \in \Delta^I \), \( \exists ! (C_i(z))_{i \in I} \in \mathcal{C} \) satisfying equations (1).
- **Aim:** find \( z \in \Delta^I \) such that \( \mathbb{E}_{Q(z)} [C_i(z)] = 0 \) holds for all \( i \in I \).

**Theorem**

1. In a Nash equilibrium, \( \mathbb{E}_{Q(z^\diamond)} [C_i(z^\diamond)] = 0 \) holds \( \forall i \in I \).
   
2. Let \( z^\diamond \in \Delta^I \) be such that \( \mathbb{E}_{Q(z^\diamond)} [C_i(z^\diamond)] = 0 \) holds \( \forall i \in I \). Then, \((Q^\diamond, (C_i^\diamond)_{i \in I})\) defined by (1) and (2) for \( z = z^\diamond \) is a Nash equilibrium.

**Theorem**

If \( I = \{0, 1\} \), there exists a unique \( z^\diamond \in \Delta^I \equiv \mathbb{R} \) with \( \mathbb{E}_{Q(z^\diamond)} [C_i(z^\diamond)] = 0, \forall i \in I \).
Existence (and uniqueness) of Nash equilibria?

In search of equilibrium

Parametrise candidate optimal contracts in

$$\Delta^I := \{(z_i)_{i \in I} \in \mathbb{R}^I \mid \sum_{i \in I} z_i = 0\} \equiv \mathbb{R}^n \quad (\text{where } n = \# I - 1).$$

- For all $z \in \Delta^I$, $\exists! (C_i(z))_{i \in I} \in \mathcal{C}$ satisfying equations (1).
- **Aim**: find $z \in \Delta^I$ such that $\mathbb{E}_{Q(z)}[C_i(z)] = 0$ holds for all $i \in I$.

**Theorem**

1. In a Nash equilibrium, $\mathbb{E}_{Q(z^\diamond)}[C_i(z^\diamond)] = 0$ holds $\forall i \in I$.
2. Let $z^\diamond \in \Delta^I$ be such that $\mathbb{E}_{Q(z^\diamond)}[C_i(z^\diamond)] = 0$ holds $\forall i \in I$. Then, $(Q^\diamond, (C_i^\diamond)_{i \in I})$ defined by (1) and (2) for $z = z^\diamond$ is a Nash equilibrium.

**Theorem**

If $I = \{0, 1\}$, there exists a unique $z^\diamond \in \Delta^I \equiv \mathbb{R}$ with $\mathbb{E}_{Q(z^\diamond)}[C_i(z^\diamond)] = 0$, $\forall i \in I$. 

M. Anthropelos (Un. Piraeus)
Equilibrium in Risk Sharing Games
Banff 2014 15 / 24
Existence (and uniqueness) of Nash equilibria?

In search of equilibrium

Parametrise candidate optimal contracts in

\[ \Delta^I := \{(z_i)_{i \in I} \in \mathbb{R}^I \mid \sum_{i \in I} z_i = 0\} \equiv \mathbb{R}^n \quad (\text{where } n = \#I - 1). \]

- For all \( z \in \Delta^I \), \( \exists! (C_i(z))_{i \in I} \in \mathcal{C} \) satisfying equations (1).
- **Aim:** find \( z \in \Delta^I \) such that \( \mathbb{E}_{Q(z)}[C_i(z)] = 0 \) holds for all \( i \in I \).

**Theorem**

1. In a Nash equilibrium, \( \mathbb{E}_{Q(z^\Diamond)}[C_i(z^\Diamond)] = 0 \) holds \( \forall i \in I \).
2. Let \( z^\Diamond \in \Delta^I \) be such that \( \mathbb{E}_{Q(z^\Diamond)}[C_i(z^\Diamond)] = 0 \) holds \( \forall i \in I \). Then, \( (Q^\Diamond, (C_i^\Diamond)_{i \in I}) \) defined by (1) and (2) for \( z = z^\Diamond \) is a Nash equilibrium.

**Theorem**

If \( I = \{0, 1\} \), there exists a unique \( z^\Diamond \in \Delta^I \equiv \mathbb{R} \) with \( \mathbb{E}_{Q(z^\Diamond)}[C_i(z^\Diamond)] = 0, \forall i \in I \).
Some consequences of Nash equilibrium

You trade, you lie

\[ F_i = E_i - z_i + \delta_i \log \left( 1 + \frac{C_i}{\delta_i} \right). \]

For any fixed \( i \in I \), \( F_i \sim E_i \iff C_i = 0 \).

Endogenous bounds on contracts

It holds that \( C_i > -\delta_i \) for all \( i \in I \). Hence,

\[ -\delta_i < C_i < (n - 1)\delta + \delta_i, \quad \forall i \in I. \]

[Contrast with A-D equilibrium.]

Aggregate loss of efficiency (in monetary terms)

\[ \sum_{i \in I} U_i(E_i + C_i^*) - \sum_{i \in I} U_i(E_i + C_i) = -\delta \log E_{Q_0} \left[ \prod_{i \in I} \left( 1 + \frac{C_i}{\delta_i} \right)^{\delta_i/\delta} \right] \geq 0. \]

No loss of efficiency \iff \( C_i^* = 0, \forall i \in I \) \iff \( C_i = 0, \forall i \in I \).
Some consequences of Nash equilibrium

You trade, you lie

\[ F_i^\diamond = E_i - z_i^\diamond + \delta_i \log \left( 1 + \frac{C_i^\diamond}{\delta_{-i}} \right). \]

- For any fixed \( i \in I \), \( F_i^\diamond \sim E_i \iff C_i^\diamond = 0. \)

Endogenous bounds on contracts

It holds that \( C_i^\diamond > -\delta_{-i} \) for all \( i \in I \). Hence,

\[ -\delta_{-i} < C_i^\diamond < (n-1)\delta + \delta_i, \quad \forall i \in I. \] [Contrast with A-D equilibrium.]

Aggregate loss of efficiency (in monetary terms)

\[
\sum_{i \in I} U_i(E_i + C_i^\ast) - \sum_{i \in I} U_i(E_i + C_i^\diamond) = -\delta \log E_{Q^\diamond} \left[ \prod_{i \in I} \left(1 + \frac{C_i^\diamond}{\delta_{-i}}\right)^{\delta_i/\delta} \right] \geq 0.
\]

No loss of efficiency \( \iff \) \( C_i^\ast = 0, \forall i \in I \iff C_i^\diamond = 0, \forall i \in I. \)
Some consequences of Nash equilibrium

You trade, you lie

\[ F_i^\diamond = E_i - z_i^\diamond + \delta_i \log \left( 1 + \frac{C_i^\diamond}{\delta_i} \right). \]

- For any fixed \( i \in I \), \( F_i^\diamond \sim E_i \iff C_i^\diamond = 0. \)

Endogenous bounds on contracts

It holds that \( C_i^\diamond > -\delta_i \) for all \( i \in I \). Hence,

\[ -\delta_i < C_i^\diamond < (n-1)\delta + \delta_i, \quad \forall i \in I. \quad \text{[Contrast with A-D equilibrium.]} \]

Aggregate loss of efficiency (in monetary terms)

\[ \sum_{i \in I} U_i(E_i + C_i^*) - \sum_{i \in I} U_i(E_i + C_i^\diamond) = -\delta \log E_{Q^\diamond} \left[ \prod_{i \in I} \left( 1 + \frac{C_i^\diamond}{\delta_i} \right)^{\delta_i/\delta} \right] \geq 0. \]

No loss of efficiency  \iff  \( C_i^* = 0, \ \forall i \in I \)  \iff  \( C_i^\diamond = 0, \ \forall i \in I. \)
An example of symmetric inefficiency

Two-person symmetric game

- \( I = \{0, 1\} \).
- \( \delta_0 = 1 = \delta_1 \).
- \( E_0 = \sigma X = -E_1 \), where \( \sigma > 0 \) and \( X \) has standard normal distribution.

Arrow-Debreu equilibrium

- \( C_0^* = E_1 = -E_0 \), \( C_1^* = E_0 = -E_1 \); no risk after transaction.

Nash equilibrium

Contract \( C_0^\diamond \) for agent 0 satisfies \(-1 < C_0^\diamond < 1\) and

\[
C_0^\diamond + \frac{1}{2} \log \left( \frac{1 + C_0^\diamond}{1 - C_0^\diamond} \right) = -E_0 \quad (= -\sigma X).
\]

Same monetary loss for both agents, becoming enormous when \( \sigma \to \infty \).

- When \( \sigma \to \infty \), \( C_0^\diamond \to -\text{sign}(X) \).
An example of symmetric inefficiency

Two-person symmetric game

- $I = \{0, 1\}$.
- $\delta_0 = 1 = \delta_1$.
- $E_0 = \sigma X = -E_1$, where $\sigma > 0$ and $X$ has standard normal distribution.

Arrow-Debreu equilibrium

- $C_0^* = E_1 = -E_0$, $C_1^* = E_0 = -E_1$; no risk after transaction.

Nash equilibrium

Contract $C_0^\diamond$ for agent 0 satisfies $-1 < C_0^\diamond < 1$ and

$$C_0^\diamond + \frac{1}{2} \log \left( \frac{1 + C_0^\diamond}{1 - C_0^\diamond} \right) = -E_0 = -\sigma X.$$

Same monetary loss for both agents, becoming enormous when $\sigma \to \infty$.

- When $\sigma \to \infty$, $C_0^\diamond \to -\text{sign}(X)$. 
An example of symmetric inefficiency

Two-person symmetric game

- \( I = \{0, 1\} \).
- \( \delta_0 = 1 = \delta_1 \).
- \( E_0 = \sigma X = -E_1 \), where \( \sigma > 0 \) and \( X \) has standard normal distribution.

Arrow-Debreu equilibrium

- \( C_0^* = E_1 = -E_0, \ C_1^* = E_0 = -E_1 \); no risk after transaction.

Nash equilibrium

Contract \( C_0^{\diamond} \) for agent 0 satisfies \( -1 < C_0^{\diamond} < 1 \) and

\[
C_0^{\diamond} + \frac{1}{2} \log \left( \frac{1 + C_0^{\diamond}}{1 - C_0^{\diamond}} \right) = -E_0 \ (= -\sigma X).
\]

Same monetary loss for both agents, becoming enormous when \( \sigma \to \infty \).
- When \( \sigma \to \infty \), \( C_0^{\diamond} \to -\text{sign}(X) \).
Outline

1. Risk sharing and Arrow-Debreu equilibrium
2. Agent’s best endowment response
3. Nash equilibria in risk sharing
4. Extreme risk tolerance
5. Conclusive remarks & open questions
A sequence of markets

Set-up and notation

- Two agents: $I = \{0, 1\}$.
- A sequence of markets, indexed by $m \in \mathbb{N}$.
- $\delta^m_1 \equiv \delta_1 \in (0, \infty)$ for all $m \in \mathbb{N}$, whereas $\lim_{m \to \infty} \delta^m_0 = \infty$.
- $E_0$ and $E_1$ fixed.

Arrow-Debreu limit

- Limiting valuation measure $Q^\infty, \ast = \mathbb{P}$.
- Limiting contracts: $C^\infty, \ast_0$ and $C^\infty, \ast_1 = -C^\infty, \ast_0$, with
  \[ C^\infty, \ast_0 = E_1 - \mathbb{E}[E_1]. \]
- Limiting utility gain (in monetary terms): with
  \[ u^\infty, \ast_i := \lim_{m \to \infty} \left( \mathbb{U}^m_i (E_i + C^m_i, \ast) - \mathbb{U}^m_i (E_i) \right), \quad \forall i \in \{0, 1\}, \]
  it holds that
  \[ u^\infty, \ast_0 = 0, \quad u^\infty, \ast_1 = \mathbb{E}[E_1] - \mathbb{U}_1(E_1). \]
A sequence of markets

Set-up and notation

- Two agents: \( I = \{0, 1\} \).
- A sequence of markets, indexed by \( m \in \mathbb{N} \).
- \( \delta_1^m \equiv \delta_1 \in (0, \infty) \) for all \( m \in \mathbb{N} \), whereas \( \lim_{m \to \infty} \delta_0^m = \infty \).
- \( E_0 \) and \( E_1 \) fixed.

Arrow-Debreu limit

- Limiting valuation measure \( \mathbb{Q}^\infty,* = \mathbb{P} \).
- Limiting contracts: \( C_0^\infty,* \) and \( C_1^\infty,* = -C_0^\infty,* \), with
  \[
  C_0^\infty,* = E_1 - \mathbb{E}[E_1].
  \]
- Limiting utility gain (in monetary terms): with
  \[
  u_i^\infty,* := \lim_{m \to \infty} \left( \mathbb{U}_i^m (E_i + C_i^m,*)) - \mathbb{U}_i^m (E_i) \right), \quad \forall i \in \{0, 1\},
  \]
  it holds that
  \[
  u_0^\infty,* = 0, \quad u_1^\infty,* = \mathbb{E}[E_1] - \mathbb{U}_1(E_1).
  \]
A sequence of markets

Set-up and notation

- Two agents: \( I = \{0, 1\} \).
- A sequence of markets, indexed by \( m \in \mathbb{N} \).
- \( \delta_1^m \equiv \delta_1 \in (0, \infty) \) for all \( m \in \mathbb{N} \), whereas \( \lim_{m \to \infty} \delta_0^m = \infty \).
- \( E_0 \) and \( E_1 \) fixed.

Arrow-Debreu limit

- Limiting valuation measure \( Q_{\infty,*} = \mathbb{P} \).
- Limiting contracts: \( C_{0,\infty,*} \) and \( C_{1,\infty,*} = -C_{0,\infty,*} \), with
  \[
  C_{0,\infty,*} = E_1 - \mathbb{E}[E_1].
  \]
- Limiting utility gain (in monetary terms): with
  \[
  u_{i,\infty,*} := \lim_{m \to \infty} \left( \mathbb{U}_i^m (E_i + C_i^m) - \mathbb{U}_i^m (E_i) \right), \quad \forall i \in \{0, 1\},
  \]
  it holds that
  \[
  u_{0,\infty,*} = 0, \quad u_{1,\infty,*} = \mathbb{E}[E_1] - \mathbb{U}_1(E_1).
  \]
A sequence of markets

Set-up and notation

- Two agents: \( I = \{0, 1\} \).
- A sequence of markets, indexed by \( m \in \mathbb{N} \).
- \( \delta_1^m \equiv \delta_1 \in (0, \infty) \) for all \( m \in \mathbb{N} \), whereas \( \lim_{m \to \infty} \delta_0^m = \infty \).
- \( E_0 \) and \( E_1 \) fixed.

Arrow-Debreu limit

- Limiting valuation measure \( Q^\infty, \ast \equiv P \).
- Limiting contracts: \( C_0^\infty, \ast \) and \( C_1^\infty, \ast = -C_0^\infty, \ast \), with
  \[
  C_0^\infty, \ast = E_1 - \mathbb{E}[E_1].
  \]
- Limiting utility gain (in monetary terms): with
  \[
  u_i^\infty, \ast := \lim_{m \to \infty} \left( \mathbb{U}_i^m (E_i + C_i^m, \ast ) - \mathbb{U}_i^m (E_i) \right), \quad \forall i \in \{0, 1\},
  \]
  it holds that
  \[
  u_0^\infty, \ast = 0, \quad u_1^\infty, \ast = \mathbb{E}[E_1] - \mathbb{U}_1(E_1).
  \]
**Game limit**

**Limiting contracts and valuation**

- Limiting Nash-equilibrium contract $C_0^{\infty,\diamond}$ for agent 0 satisfies

$$C_0^{\infty,\diamond} + \delta_1 \log \left( 1 + \frac{C_0^{\infty,\diamond}}{\delta_1} \right) = z^{\infty,\diamond} + E_1,$$

where $z^{\infty,\diamond} \in \mathbb{R}$ is such that $\mathbb{E} \left[ (1 + C_0^{\infty,\diamond}/\delta_1)^{-1} \right] = 1$. Furthermore,

$$dQ^{\infty,\diamond} = (1 + C_0^{\infty,\diamond}/\delta_1)^{-1} dP.$$

- $F_1^{\infty,\diamond} \sim E_1$. On the other hand, $F_m^{\infty,\diamond}$ is $O_p(\delta_m)$ as $m \to \infty$.

**Limiting utility gain/loss (in monetary terms)**

With $u_i^{\infty,\diamond} := \lim_{m \to \infty} (U_i^m (E_i + C_i^{m,\diamond}) - U_i^m (E_i))$ for $i \in \{0, 1\}$, it holds that

$$u_0^{\infty,\diamond} = u_0^{\infty,*} + \frac{1}{\delta_1} \text{Var}_{Q^{\infty,\diamond}} (C_0^{\infty,\diamond}),$$

$$u_1^{\infty,\diamond} = u_1^{\infty,*} - \frac{1}{\delta_1} \text{Var}_{Q^{\infty,\diamond}} (C_0^{\infty,\diamond}) - \delta_1 \mathcal{H} (P | Q^{\infty,\diamond}).$$
Game limit

Limiting contracts and valuation

- Limiting Nash-equilibrium contract $C_0^{\infty,\Diamond}$ for agent 0 satisfies

$$C_0^{\infty,\Diamond} + \delta_1 \log \left( 1 + \frac{C_0^{\infty,\Diamond}}{\delta_1} \right) = z^{\infty,\Diamond} + E_1,$$

where $z^{\infty,\Diamond} \in \mathbb{R}$ is such that $\mathbb{E} \left[ (1 + C_0^{\infty,\Diamond}/\delta_1)^{-1} \right] = 1$. Furthermore,

$$dQ^{\infty,\Diamond} = (1 + C_0^{\infty,\Diamond}/\delta_1)^{-1} dP.$$

- $F_1^{\infty,\Diamond} \sim E_1$. On the other hand, $F_0^{m,\Diamond}$ is $O_p(\delta_0^m)$ as $m \to \infty$.

Limiting utility gain/loss (in monetary terms)

With $u_i^{\infty,\Diamond} := \lim_{m \to \infty} \left( \mathbb{U}_i^m (E_i + C_i^{m,\Diamond}) - \mathbb{U}_i^m (E_i) \right)$ for $i \in \{0, 1\}$, it holds that

$$u_0^{\infty,\Diamond} = u_0^{\infty,*} + \frac{1}{\delta_1} \text{Var}_{Q^{\infty,\Diamond}} (C_0^{\infty,\Diamond}),$$

$$u_1^{\infty,\Diamond} = u_1^{\infty,*} - \frac{1}{\delta_1} \text{Var}_{Q^{\infty,\Diamond}} (C_0^{\infty,\Diamond}) - \delta_1 \mathcal{H} (P | Q^{\infty,\Diamond}).$$
Game limit

Limiting contracts and valuation

- Limiting Nash-equilibrium contract $C_0^{\infty, \Diamond}$ for agent 0 satisfies

$$C_0^{\infty, \Diamond} + \delta_1 \log \left(1 + \frac{C_0^{\infty, \Diamond}}{\delta_1}\right) = z^{\infty, \Diamond} + E_1,$$

where $z^{\infty, \Diamond} \in \mathbb{R}$ is such that $\mathbb{E}\left[\left(1 + \frac{C_0^{\infty, \Diamond}}{\delta_1}\right)^{-1}\right] = 1$. Furthermore,

$$dQ^{\infty, \Diamond} = \left(1 + \frac{C_0^{\infty, \Diamond}}{\delta_1}\right)^{-1} d\mathbb{P}.$$

- $F_1^{\infty, \Diamond} \sim E_1$. On the other hand, $F_0^{m, \Diamond}$ is $O_p(\delta_0^m)$ as $m \to \infty$.

Limiting utility gain/loss (in monetary terms)

With $u_i^{\infty, \Diamond} := \lim_{m \to \infty} \left(U_i^m (E_i + C_i^{m, \Diamond}) - U_i^m (E_i)\right)$ for $i \in \{0, 1\}$, it holds that

$$u_0^{\infty, \Diamond} = u_0^{\infty, *} + \frac{1}{\delta_1} \text{Var}_{Q^{\infty, \Diamond}} \left(C_0^{\infty, \Diamond}\right),$$

$$u_1^{\infty, \Diamond} = u_1^{\infty, *} - \frac{1}{\delta_1} \text{Var}_{Q^{\infty, \Diamond}} \left(C_0^{\infty, \Diamond}\right) - \delta_1 \mathcal{H}(\mathbb{P} | Q^{\infty, \Diamond}).$$
Both agents close to risk neutrality

Set-up

- Two agents: \( I = \{0, 1\} \).
- A sequence of markets, indexed by \( m \in \mathbb{N} \).
- Both \( \lim_{m \to \infty} \delta_0^m = \infty \), \( \lim_{m \to \infty} \delta_1^m = \infty \), but...
- \( \lambda_0 \) and \( \lambda_1 \) fixed, not depending on \( m \in \mathbb{N} \).
- \( E_0 \) and \( E_1 \) fixed, not depending on \( m \in \mathbb{N} \).

Limits

- Limiting valuation measures: \( Q_\infty,* = P = Q_\infty,* \).
- Limiting contracts are:

\[
C_{0,*}^\infty = \lambda_0 E_1 - \lambda_1 E_0 - \mathbb{E} [\lambda_0 E_1 - \lambda_1 E_0],
\]

\[
C_{0,*} = \frac{C_{0,*}^\infty}{2}.
\]

There is decrease in trading volume.
Both agents close to risk neutrality

**Set-up**
- Two agents: \( I = \{0, 1\} \).
- A sequence of markets, indexed by \( m \in \mathbb{N} \).
- Both \( \lim_{m \to \infty} \delta_m^0 = \infty \), \( \lim_{m \to \infty} \delta_m^1 = \infty \), but . . .
- \( \lambda_0 \) and \( \lambda_1 \) fixed, not depending on \( m \in \mathbb{N} \).
- \( E_0 \) and \( E_1 \) fixed, not depending on \( m \in \mathbb{N} \).

**Limits**
- Limiting valuation measures: \( Q^\infty,* = P = Q^\infty,\diamond \).
- Limiting contracts are:
  \[
  C_0^\infty,* = \lambda_0 E_1 - \lambda_1 E_0 - \mathbb{E} [\lambda_0 E_1 - \lambda_1 E_0],
  
  C_0^\infty,\diamond = \frac{C_0^\infty,*}{2}.
  \]

There is *decrease in trading volume.*
Both agents close to risk neutrality

Set-up

- Two agents: \( I = \{0, 1\} \).
- A sequence of markets, indexed by \( m \in \mathbb{N} \).
- Both \( \lim_{m \to \infty} \delta_m^0 = \infty \), \( \lim_{m \to \infty} \delta_m^1 = \infty \), but...
- \( \lambda_0 \) and \( \lambda_1 \) fixed, not depending on \( m \in \mathbb{N} \).
- \( E_0 \) and \( E_1 \) fixed, not depending on \( m \in \mathbb{N} \).

Limits

- Limiting valuation measures: \( Q^{\infty,*} = P = Q^{\infty,\diamond} \).
- Limiting contracts are:

\[
C^{\infty,*}_0 = \lambda_0 E_1 - \lambda_1 E_0 - \mathbb{E} [\lambda_0 E_1 - \lambda_1 E_0], \quad C^{\infty,\diamond}_0 = \frac{C^{\infty,*}_0}{2}.
\]

There is decrease in trading volume.
Outline

1. Risk sharing and Arrow-Debreu equilibrium
2. Agent’s best endowment response
3. Nash equilibria in risk sharing
4. Extreme risk tolerance
5. Conclusive remarks & open questions
Conclusive remarks & open questions

Conclusive remarks

- This work attempts to introduce strategic behaviour in the risk sharing literature.
- Such strategic behaviour gives an endogenous explanation of the risk sharing inefficiency and security mispricing that occur in markets with few agents.
- Agents trading in Nash equilibrium never report their true risk exposure.
- In symmetric games, every agent suffers loss of utility as compared to the Arrow-Debreu equilibrium one.
- Strategic games benefit agents with high risk tolerance.

Ahead?

- Existence (and uniqueness?) for more than two players.
- Strategic behaviour when trading given securities.
- Other risk-sharing rules?
- Include risk tolerance as control?
- Dynamic framework?
Conclusive remarks & open questions

Conclusive remarks

- This work attempts to introduce strategic behaviour in the risk sharing literature.
- Such strategic behaviour gives an endogenous explanation of the risk sharing inefficiency and security mispricing that occur in markets with few agents.
- Agents trading in Nash equilibrium never report their true risk exposure.
- In symmetric games, every agent suffers loss of utility as compared to the Arrow-Debreu equilibrium one.
- Strategic games benefit agents with high risk tolerance.

Ahead?

- Existence (and uniqueness?) for more than two players.
- Strategic behaviour when trading given securities.
- Other risk-sharing rules?
- Include risk tolerance as control?
- Dynamic framework?
The End

Thank you for your attention!

For a preprint, email to
anthropel@unipi.gr