Equilibrium in risk-sharing games

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Motivation

- Financial agents sharing their risky position by designing new financial contracts in a mutually beneficial way.
- Such risk sharing involves only a small number of agents. Each agent can influence the equilibrium sharing; $\rightarrow$ not a cooperative equilibrium.
- Agents' strategic behaviour in risk sharing should be introduced.

We ask:

✓ *How much risk should an agent share?* (Best response problem)

✓ *How and at which point the market equilibrate?* (Nash equilibrium)

✓ *Do certain agents benefit from the game?* (Equilibria comparison)

(Very) short list of related literature

- On optimal risk sharing: Seminal works of Borch ['62, '68] and Wilson ['68]. See also Duffie & Rahi ['95], Barrieu & El Karoui ['04, '05], Jouini, Schachermayer & Touzi ['08] etc.
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Outline

1. Risk sharing and Arrow-Debreu equilibrium
2. Agent's best endowment response
3. Nash equilibria in risk sharing
4. Extreme risk tolerance
5. Conclusive remarks & open questions

Outline

Agents and preferences

Static probability model

- $\mathbb{L}^0 \equiv \mathbb{L}^0(\Omega, \mathcal{F}, \mathbb{P})$: *discounted* future financial positions.
- $I = \{0, \ldots, n\}$: index set of $n+1$ economic agents.

Preferences

- Agents' risk preferences modelled via *monetary* utility functionals:

$$\mathbb{L}^0 \ni X \mapsto \mathbb{U}_i(X) := -\delta_i \log\left(\mathbb{E}\left[\exp\left(-\frac{X}{\delta_i}\right)\right]\right) \in [-\infty, \infty).$$

- Define the aggregate risk tolerance

$$\delta := \sum_{i \in I} \delta_i,$$

as well as

$$\lambda_i := \frac{\delta_i}{\delta}, \quad \delta_{-i} := \delta - \delta_i, \quad \forall i \in I.$$

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Endowments and Contracts

Endowments

- $E_i \in \mathbb{L}^0$: random endowment (risky position) of agent $i \in I$.
- Aggregate endowment:

$$E := \sum_{i \in I} E_i.$$

- **Standing assumption** enforced throughout: $(E_i)_{i \in I} \in \mathcal{E}$; in effect,

$$\mathbb{U}_i(E_i) > -\infty, \quad \forall i \in I.$$

Sharing via contracts

$$\mathcal{C} := \left\{ (C_i)_{i \in I} \in \left(\mathbb{L}^0\right)^I \;\middle|\; \sum_{i \in I} C_i = 0 \right\}.$$

$\rightarrow$ After sharing, position of agent $i \in I$ is $E_i + C_i$.

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Complete market equilibrium

Arrow-Debreu equilibrium

Valuation probability $\mathbb{Q}^*$ (equivalent to $\mathbb{P}$) and contracts $(C_i^*)_{i \in I} \in \mathcal{C}$ such that:

- $\mathbb{E}_{\mathbb{Q}^*}\left[C_i^*\right] = 0, \ \forall i \in I$.
- $\mathbb{U}_i(E_i + C_i) \leq \mathbb{U}_i(E_i + C_i^*)$, $\forall i \in I$ and $C_i \in \mathbb{L}^0$ with $\mathbb{E}_{\mathbb{Q}^*}\left[C_i\right] \leq 0$.

Theorem (Borch '62)

A unique Arrow-Debreu equilibrium exists; in fact, $\mathrm{d}\mathbb{Q}^*/\mathrm{d}\mathbb{P} \propto \exp\left(-E/\delta\right)$ and

$$C_i^* := \lambda_i E - E_i - \mathbb{E}_{\mathbb{Q}^*}\left[\lambda_i E - E_i\right], \quad \forall i \in I.$$

Aggregate monetary utility in Arrow-Debreu equilibrium

$(C_i^*)_{i \in I}$ is a maximiser of $\mathcal{C} \ni (C_i)_{i \in I} \mapsto \sum_{i \in I} \mathbb{U}_i(E_i + C_i)$; furthermore,

$$\textstyle\sum_{i \in I} \mathbb{U}_i(E_i + C_i^*) = -\delta \log \mathbb{E}\left[\exp\left(-E/\delta\right)\right] \geq \sum_{i \in I} \mathbb{U}_i(E_i).$$

$\rightarrow$ "$\geq$" above is "$=$" $\quad \Longleftrightarrow \quad C_i^* = 0, \forall i \in I$.

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Reported endowments

Agents may have motive to report different endowments than their actual ones.

What if instead of $(E_i)_{i \in I} \in \mathcal{E}$, agents choose to report $(F_i)_{i \in I} \in \mathcal{E}$?

- With $F := \sum_{i \in I} F_i$, the valuation measure $\mathbb{Q}^F$ is such that
$$\mathrm{d}\mathbb{Q}^F / \mathrm{d}\mathbb{P} \propto \exp\left(-F/\delta\right).$$
- Leads to risk-sharing with contracts
$$\begin{aligned} C_i &= \lambda_i F - F_i - \mathbb{E}_{\mathbb{Q}^F}\left[\lambda_i F - F_i\right] \\ &= \lambda_i F_{-i} - \lambda_{-i} F_i - \mathbb{E}_{\mathbb{Q}^{F_{-i}+F_i}}\left[\lambda_i F_{-i} - \lambda_{-i} F_i\right], \quad \forall i \in I, \qquad (\star) \end{aligned}$$

Stage 1: Agents agree on the sharing rules of the *reported* endowments.

Revealed endowments via valuation measure and contracts

Given $\mathbb{Q}$ and $(C_i)_{i \in I} \in \mathcal{C}$ such that $\mathbb{E}_{\mathbb{Q}}\left[C_i\right] = 0$, $\forall i \in I$
$\exists (F_i)_{i \in I}$ (unique up to cash translation) such that

$$\mathbb{Q} = \mathbb{Q}^F \quad \text{and} \quad (C_i)_{i \in I} \quad \text{are given by } (\star).$$

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Best endowment response: the problem

Consider the position of agent $i \in I$. Given

- the agreed mechanism that produces the optimal sharing contracts; and
- the endowment F_{-i} reported by the rest n agents in $I \setminus \{i\}$,

a natural question is:

Which random quantity should agent $i \in I$ report as actual endowment?

Response function

Let F_{-i} given. The **response function** of agent $i \in I$ is

$$\mathbb{V}_i(F_i; F_{-i}) := \mathbb{U}_i \left(E_i + \lambda_i F_{-i} - \lambda_{-i} F_i - \mathbb{E}_{\mathbb{Q}^{F_{-i}+F_i}} \left[\lambda_i F_{-i} - \lambda_{-i} F_i \right] \right).$$

- $\mathbb{V}_i(F_i + c; F_{-i}) = \mathbb{V}_i(F_i; F_{-i})$ holds for all $c \in \mathbb{R}$.
- $\mathbb{V}_i(\cdot; F_{-i})$ is *not* concave in general.

Best response

For given F_{-i}, we seek F_i^r such that

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Best endowment response: results

Proposition (Necessary and sufficient conditions for optimality)

Let $i \in I$, F_{-i} and F_i^{r} given. The following are equivalent:

1. $\mathbb{V}_i(F_i^{\mathrm{r}}; F_{-i}) = \sup_{F_i} \mathbb{V}_i(F_i; F_{-i})$.
2. $C_i^{\mathrm{r}} := \lambda_i F_{-i} - \lambda_{-i} F_i^{\mathrm{r}} - \mathbb{E}_{\mathbb{Q}^{F_{-i}+F_i^{\mathrm{r}}}} [\lambda_i F_{-i} - \lambda_{-i} F_i^{\mathrm{r}}]$ is such that

$$\delta \frac{C_i^{\mathrm{r}}}{\delta_{-i}} + \delta_i \log \left(1 + \frac{C_i^{\mathrm{r}}}{\delta_{-i}} \right) = z_i^{\mathrm{r}} - E_i + \delta_i \frac{F_{-i}}{\delta_{-i}},$$

(note the *a-priori* necessary bound $C_i^{\mathrm{r}} > -\delta_{-i}$) and $z_i^{\mathrm{r}} \in \mathbb{R}$ is such that

$$z_i^{\mathrm{r}} = \mathbb{U}_i(E_i + C_i^{\mathrm{r}}) - \mathbb{U}_i \left(\frac{\delta_i}{\delta_{-i}} (F_{-i} - C_i^{\mathrm{r}}) \right).$$

(1) $\Rightarrow$ (2): 1st-order conditions. $\mathbb{V}_i(\cdot; F_{-i})$ is not concave: (2) $\Rightarrow$ (1) is tricky.

Theorem

There exists unique (up to constants) F_i^{r} s.t. $\mathbb{V}_i(F_i^{\mathrm{r}}; F_{-i}) = \sup_{F_i} \mathbb{V}_i(F_i; F_{-i})$.

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(1) $\Rightarrow$ (2): 1st-order conditions. $\mathbb{V}_i(\cdot; F_{-i})$ is not concave: (2) $\Rightarrow$ (1) is tricky.

Theorem

There exists unique (up to constants) F_i^{r} s.t. $\mathbb{V}_i(F_i^{\mathrm{r}}; F_{-i}) = \sup_{F_i} \mathbb{V}_i(F_i; F_{-i})$.

Best endowment response: results

Proposition (Necessary and sufficient conditions for optimality)

Let $i \in I$, F_{-i} and F_i^{r} given. The following are equivalent:

1. $\mathbb{V}_i(F_i^{\mathrm{r}}; F_{-i}) = \sup_{F_i} \mathbb{V}_i(F_i; F_{-i})$.
2. $C_i^{\mathrm{r}} := \lambda_i F_{-i} - \lambda_{-i} F_i^{\mathrm{r}} - \mathbb{E}_{\mathbb{Q}^{F_{-i}+F_i^{\mathrm{r}}}}[\lambda_i F_{-i} - \lambda_{-i} F_i^{\mathrm{r}}]$ is such that

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Outline

Nash Equilibrium

Stage 2

- All agents have same strategic behaviour.
- Given the agreed risk sharing rules (stage 1), agents negotiate the contracts they are going to trade and the valuation measure.

Definition

A valuation measure $\mathbb{Q}^\diamond$ and a collection of contracts $(C_i^\diamond)_{i\in I} \in \mathcal{C}$ will be called a **game (Nash) equilibrium** if

$$\mathbb{V}_i\left(F_i^\diamond; F_{-i}^\diamond\right) = \sup_{F_i} \mathbb{V}_i\left(F_i; F_{-i}^\diamond\right), \quad \forall i \in I,$$

where $(F_i^\diamond)_{i\in I}$ are the corresponding revealed endowments, given implicitly by

$$\frac{\mathrm{d}\mathbb{Q}^\diamond}{\mathrm{d}\mathbb{P}} \propto \exp\left(-F^\diamond/\delta\right)$$

and

$$C_i^\diamond = \lambda_i F^\diamond - F_i^\diamond - \mathbb{E}_{\mathbb{Q}^\diamond}\left[\lambda_i F^\diamond - F_i^\diamond\right].$$

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Necessary and sufficient conditions for Nash equilibrium

Theorem

For given $\mathbb{Q}^\diamond$ and $(C_i^\diamond)_{i\in I} \in \mathcal{C}$, the following conditions are equivalent:

- $(\mathbb{Q}^\diamond, (C_i^\diamond)_{i\in I})$ is a Nash equilibrium.
- 1. $C_i^\diamond > -\delta_{-i}$, and there exists $z^\diamond \equiv (z_i^\diamond)_{i\in I} \in \mathbb{R}^I$ with $\sum_{i\in I} z_i^\diamond = 0$ such that
  $$C_i^\diamond + \delta_i \log\left(1 + \frac{C_i^\diamond}{\delta_{-i}}\right) = z_i^\diamond + C_i^* + \frac{\delta_i}{\delta}\sum_{j\in I}\left(1 + \delta_j \frac{C_j^\diamond}{\delta_{-j}}\right), \quad \forall i \in I. \qquad (1)$$
  2. $\mathbb{Q}^\diamond$ is such that
  $$\frac{\mathrm{d}\mathbb{Q}^\diamond}{\mathrm{d}\mathbb{Q}^*} \propto \prod_{j\in I}\left(1 + \frac{C_j^\diamond}{\delta_{-j}}\right)^{\delta_j/\delta}. \qquad (2)$$
  3. $\mathbb{E}_{\mathbb{Q}^\diamond}[C_i^\diamond] = 0, \quad \forall i \in I.$

Existence (and uniqueness) of Nash equilibria?

In search of equilibrium

Parametrise candidate optimal contracts in

$$\Delta^I := \left\{ (z_i)_{i \in I} \in \mathbb{R}^I \mid \sum_{i \in I} z_i = 0 \right\} \equiv \mathbb{R}^n \quad \text{(where } n = \#I - 1\text{)}.$$

- For all $z \in \Delta^I$, $\exists!$ $(C_i(z))_{i \in I} \in \mathcal{C}$ satisfying equations (1).
- Aim: find $z \in \Delta^I$ such that $\mathbb{E}_{\mathbb{Q}(z)}[C_i(z)] = 0$ holds for all $i \in I$.

Theorem

1. In a Nash equilibrium, $\mathbb{E}_{\mathbb{Q}(z^\diamond)}[C_i(z^\diamond)] = 0$ holds $\forall i \in I$.
2. Let $z^\diamond \in \Delta^I$ be such that $\mathbb{E}_{\mathbb{Q}(z^\diamond)}[C_i(z^\diamond)] = 0$ holds $\forall i \in I$. Then, $(\mathbb{Q}^\diamond, (C_i^\diamond)_{i \in I})$ defined by (1) and (2) for $z = z^\diamond$ is a Nash equilibrium.

Theorem

If $I = \{0, 1\}$, there exists a unique $z^\diamond \in \Delta^I \equiv \mathbb{R}$ with $\mathbb{E}_{\mathbb{Q}(z^\diamond)}[C_i(z^\diamond)] = 0$, $\forall i \in I$.

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Some consequences of Nash equilibrium

You trade, you lie

$$F_i^\diamond = E_i - z_i^\diamond + \delta_i \log\left(1 + \frac{C_i^\diamond}{\delta_{-i}}\right).$$

- For any fixed $i \in I$, $\quad F_i^\diamond \sim E_i \quad \Longleftrightarrow \quad C_i^\diamond = 0.$

Endogenous bounds on contracts

It holds that $C_i^\diamond > -\delta_{-i}$ for all $i \in I$. Hence,

$$-\delta_{-i} < C_i^\diamond < (n-1)\delta + \delta_i, \quad \forall i \in I. \quad \text{[Contrast with A-D equilibrium.]}$$

Aggregate loss of efficiency (in monetary terms)

$$\sum_{i \in I} \mathbb{U}_i(E_i + C_i^*) - \sum_{i \in I} \mathbb{U}_i(E_i + C_i^\diamond) = -\delta \log \mathbb{E}_{\mathbb{Q}^\diamond}\left[\prod_{i \in I}\left(1 + \frac{C_i^\diamond}{\delta_{-i}}\right)^{\delta_i/\delta}\right] \geq 0.$$

No loss of efficiency $\quad \Longleftrightarrow \quad C_i^* = 0, \ \forall i \in I \quad \Longleftrightarrow \quad C_i^\diamond = 0, \ \forall i \in I.$

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An example of symmetric inefficiency

Two-person symmetric game

- $I = \{0, 1\}$.
- $\delta_0 = 1 = \delta_1$.
- $E_0 = \sigma X = -E_1$, where $\sigma > 0$ and X has standard normal distribution.

Arrow-Debreu equilibrium

- $C_0^* = E_1 = -E_0$, $C_1^* = E_0 = -E_1$; *no* risk after transaction.

Nash equilibrium

Contract $C_0^\diamond$ for agent 0 satisfies $-1 < C_0^\diamond < 1$ and

$$C_0^\diamond + \frac{1}{2} \log \left(\frac{1 + C_0^\diamond}{1 - C_0^\diamond} \right) = -E_0 \ (= -\sigma X).$$

Same monetary loss for both agents, becoming enormous when $\sigma \to \infty$.

- When $\sigma \to \infty$, $C_0^\diamond \to -\mathrm{sign}(X)$.

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Outline

A sequence of markets

Set-up and notation

- Two agents: $I = \{0, 1\}$.
- A sequence of markets, indexed by $m \in \mathbb{N}$.
- $\delta_1^m \equiv \delta_1 \in (0, \infty)$ for all $m \in \mathbb{N}$, whereas $\lim_{m\to\infty} \delta_0^m = \infty$.
- E_0 and E_1 fixed.

Arrow-Debreu limit

- Limiting valuation measure $\mathbb{Q}^{\infty,*} = \mathbb{P}$.
- Limiting contracts: $C_0^{\infty,*}$ and $C_1^{\infty,*} = -C_0^{\infty,*}$, with

$$C_0^{\infty,*} = E_1 - \mathbb{E}\left[E_1\right].$$

- Limiting utility gain (in monetary terms): with

$$u_i^{\infty,*} := \lim_{m\to\infty} \left(\mathbb{U}_i^m\left(E_i + C_i^{m,*}\right) - \mathbb{U}_i^m\left(E_i\right)\right), \quad \forall i \in \{0, 1\},$$

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Game limit

Limiting contracts and valuation

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$$C_0^{\infty,\diamond} + \delta_1 \log\left(1 + \frac{C_0^{\infty,\diamond}}{\delta_1}\right) = z^{\infty,\diamond} + E_1,$$

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$$\mathrm{d}\mathbb{Q}^{\infty,\diamond} = \left(1 + C_0^{\infty,\diamond}/\delta_1\right)^{-1} \mathrm{d}\mathbb{P}.$$

- $F_1^{\infty,\diamond} \sim E_1$. On the other hand, $F_0^{m,\diamond}$ is $O_p(\delta_0^m)$ as $m \to \infty$.

Limiting utility gain/loss (in monetary terms)

With $u_i^{\infty,\diamond} := \lim_{m\to\infty}\left(\mathbb{U}_i^m\left(E_i + C_i^{m,\diamond}\right) - \mathbb{U}_i^m\left(E_i\right)\right)$ for $i \in \{0,1\}$, it holds that

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Both agents close to risk neutrality

Set-up

- Two agents: $I = \{0, 1\}$.
- A sequence of markets, indexed by $m \in \mathbb{N}$.
- Both $\lim_{m\to\infty} \delta_0^m = \infty$, $\lim_{m\to\infty} \delta_1^m = \infty$, but...
- λ_0 and λ_1 fixed, not depending on $m \in \mathbb{N}$.
- E_0 and E_1 fixed, not depending on $m \in \mathbb{N}$.

Limits

- Limiting valuation measures: $\mathbb{Q}^{\infty,*} = \mathbb{P} = \mathbb{Q}^{\infty,\diamond}$.
- Limiting contracts are:

$$C_0^{\infty,*} = \lambda_0 E_1 - \lambda_1 E_0 - \mathbb{E}\left[\lambda_0 E_1 - \lambda_1 E_0\right],$$
$$C_0^{\infty,\diamond} = \frac{C_0^{\infty,*}}{2}.$$

There is *decrease in trading volume*.

Both agents close to risk neutrality

Set-up

- Two agents: $I = \{0, 1\}$.
- A sequence of markets, indexed by $m \in \mathbb{N}$.
- Both $\lim_{m\to\infty} \delta_0^m = \infty$, $\lim_{m\to\infty} \delta_1^m = \infty$, but...
- λ_0 and λ_1 fixed, not depending on $m \in \mathbb{N}$.
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Conclusive remarks & open questions

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- This work attempts to introduce strategic behaviour in the risk sharing literature.
- Such strategic behaviour gives an endogenous explanation of the risk sharing inefficiency and security mispricing that occur in markets with few agents.
- Agents trading in Nash equilibrium *never* report their true risk exposure.
- In symmetric games, every agent suffers loss of utility as compared to the Arrow-Debreu equilibrium one.
- Strategic games *benefit* agents with high risk tolerance.

Ahead?

- Existence (and uniqueness?) for more than two players.
- Strategic behaviour when trading given securities.
- Other risk-sharing rules?
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The End

Thank you for your attention!

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