Some Aspects of Universal Portfolio

Tomoyuki Ichiba (UC Santa Barbara)
partially joint work with
Marcel Brod (ETH Zurich)

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On $(\Omega, \mathcal{F}, \mathbb{P})$ let us consider a model of equity market $\mathfrak{X} := (\mathfrak{X}(t) := (X_1(t), \ldots, X_n(t)), 0 \le t < \infty)$, $n \, (\ge 2)$

$$\mathrm{d}X_i(t) = X_i(t)\big(\beta_i(t)\mathrm{d}t + \sum_{k=1}^{K} \sigma_{i,k}(t)\mathrm{d}\, W_k(t)\big)\,;$$

$i = 1, \ldots, n\,, 0 \le t < \infty\,.$

- $\beta(\cdot) := (\beta_1(\cdot), \ldots, \beta_n(\cdot))$, $\sigma(\cdot) := (\sigma_{i,k}(\cdot))_{i,k}$ progressively measurable w.r.t. a right-continuous filtration $\mathbb{F} := (\mathcal{F}(\cdot))$,
- $(W_1, \ldots, W_K(\cdot))$ K-dimensional BM with $K \ge n$, adapted to augmented filtration $\mathbb{G}$ of $\mathbb{F}$
- we consider the model under integrability condition

$$\sum_{i=1}^{n} \int_0^T (|\beta_i(s)| + \alpha_{ii}(s))\mathrm{d}s < \infty \quad a.s.$$

where for every $1 \le i, j \le n$

$$\alpha_{i,j}(\cdot) := \sum_{k=1}^{K} \sigma_{i,k}(\cdot)\sigma_{j,k}(\cdot) = \frac{1}{X_i(\cdot)\, X_j(\cdot)} \cdot \frac{\mathrm{d}}{\mathrm{d}t} \langle X_i, X_j \rangle(\cdot)\,.$$

Portfolios $\pi(\cdot)$ for a small investor

Let us consider investments of a small investor who do not have power to affect market prices. Given the market model and some *initial capital* $V(0) = v$, the investor selects nonnegative, $\mathbb{G}$-progressively measurable, portfolio weights $\pi(\cdot) := (\pi_1(\cdot), \ldots, \pi_n(\cdot))'$ of the wealth process $V(\cdot)$, that is, the resulting wealth process $V(\cdot) \equiv V^{v,\pi}(\cdot)$ satisfies

$$\frac{\mathrm{d}\,V^{v,\pi}(t)}{V^{v,\pi}(t)} = \sum_{i=1}^{n} \pi_i(t) \frac{\mathrm{d}X_i(t)}{X_i(t)} = \pi'(t)[\beta(t)\mathrm{d}t + \sigma(t)\mathrm{d}\,W(t)]$$

with the *no short selling* constraints $\pi_i(t) \in \Delta_+^n$ for every $0 \le t < \infty$, where Δ_+^n is the unit simplex in dimension n, i.e.,

$$\Delta_+^n := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_i \ge 0, \quad 1 \le i \le n, \quad x_1 + \cdots + x_n = 1\}.$$

Market Portfolio

The performance of the investor's selection can be evaluated by comparisons of the wealth process $V^{v,\pi}(\cdot)$ of portfolio $\pi(\cdot)$ with the the wealth process $V^{v,\mu}(\cdot)$ of the market portfolio $\mu(\cdot) := (\mu_1(\cdot), \ldots, \mu_n(\cdot))$ with

$$V^{v,\mu}(\cdot) = \frac{v \cdot X(\cdot)}{X(0)}, \quad \mu_i(\cdot) := \frac{X_i(\cdot)}{X(\cdot)} \quad 1 \le i \le n, \text{ where}$$

$$X(\cdot) := X_1(\cdot) + \cdots + X_n(\cdot).$$

The wealth process $V^{v,\mu}(\cdot)$ reflects the growth of the whole market and here we measure the comparative performance of $V^{v,\pi}(\cdot)$ relative to $V^{v,\mu}(\cdot)$.

Constant weighted portfolios

- $\pi_i(\cdot) \equiv \theta_i$, $i = 1, \dots, n$ for some $\theta \in \Delta^n_+$.

By an application of Itô's formula, given a constant vector $\theta := (\theta_1, \dots, \theta_n) \in \Delta^n_+$, the wealth process $V^{v,\theta}(\cdot)$ of constant rebalanced portfolio $\pi_i(\cdot) \equiv \theta_i$ can be written as

$$V^{v,\theta}(\cdot) = v \cdot \prod_{i=1}^{n} \Big[\frac{X_i(\cdot)}{X_i(0)} \Big]^{\theta_i} \cdot \exp\Big(\int_0^{\cdot} \gamma^*_\theta(s) \mathrm{d}s \Big) ,$$

where $\gamma^*_\theta(\cdot)$ is the *excess growth rate* of the constant rebalanced portfolio θ defined by

$$\gamma^*_\theta(\cdot) := \frac{1}{2} \Big(\sum_{i=1}^{n} \theta_i \, \alpha_{i,i}(\cdot) - \sum_{i,j=1}^{n} \theta_i \, \alpha_{i,j}(\cdot) \, \theta_j \Big) .$$

Universal portfolios

- $\pi^u(\cdot) := (\pi_1^u(\cdot), \ldots, \pi_n^u(\cdot))$, proposed by Cover ('91), are portfolios based on the market performances defined by

$$\pi_i^u(\cdot) := \Big[\int_{\Delta_+^n} V^{v,\theta}(\cdot)\mathrm{d}\theta\Big]^{-1}\Big[\int_{\Delta_+^n} \theta_i\, V^{v,\theta}(t)\,\mathrm{d}\theta\Big] \qquad (1)$$

with the resulting wealth process $V^{v,\pi^u}(\cdot)$ given by

$$V^{v,\pi^u}(\cdot) = \Big[\int_{\Delta_+^n} \mathrm{d}\theta\Big]^{-1}\Big[\int_{\Delta_+^n} V^{v,\theta}(\cdot)\,\mathrm{d}\theta\Big]$$

where $V^{v,\theta}(\cdot)$ is the wealth process of the constant rebalanced portfolio $\theta \in \Delta_+^n$ and the integration over Δ_+^n means integration with respect to $(n-1)$ variables $(\theta_1, \ldots, \theta_{n-1})$ with $0 \le \theta_n = 1 - (\theta_1 + \cdots + \theta_{n-1}) \le 1$.

- What are comparative characteristics of Universal Portfolios?

• JAMSHIDIAN ('92) : under some weak regularity conditions the universal portfolio outperforms constant rebalanced portfolios in the long-run, and moreover, the universal portfolio is asymptotically weakly optimal.

• Let us define the cumulative excess growth process $\varphi^*(\theta, \cdot)$ of constantly rebalanced portfolio $\theta \in \Delta_+^n$:

$$\varphi^*(\theta, \cdot) := \int_0^{\cdot} \gamma_\theta^*(s) \mathrm{d}s = \frac{1}{2}\Big(\sum_{i=1}^n \theta_i \int_0^{\cdot} \alpha_{i,i}(s) \mathrm{d}s - \sum_{i,j=1}^n \theta_i \theta_j \int_0^{\cdot} \alpha_{i,j}(s) \mathrm{d}s \Big) ,$$

$\mathbb{G}$-progressively measurable, continuous process of finite variation.

The universal portfolio is the moment

$$\pi_i^u(\cdot) \;=\; \int_{\Delta_+^n} \theta_i \, \nu^{**}(\mathrm{d}\theta, \cdot)\,,$$

for $i = 1, \ldots, n$, where $\nu^{**}(\mathrm{d}\theta, \cdot)$ is a measure-valued, stochastic process defined by

$$\nu^{**}(\mathrm{d}\theta, \cdot) \;:=\; \Big[\int_{\Delta_+^n} \prod_{i=1}^{n} \Big[\frac{X_i(\cdot)}{X_i(0)} \Big]^{\theta_i} \nu^{*}(\mathrm{d}\theta, \cdot) \Big]^{-1} \prod_{i=1}^{n} \Big[\frac{X_i(\cdot)}{X_i(0)} \Big]^{\theta_i} \nu^{*}(\mathrm{d}\theta, \cdot)\,;$$

with the exponentially tilted, measure-valued process

$$\nu^{*}(\mathrm{d}\theta, \cdot) \;:=\; \Big[\int_{\Delta_+^n} \exp\big(\varphi^*(\xi, \cdot)\big) \mathrm{d}\xi \Big]^{-1} \exp\big(\varphi^*(\theta, \cdot)\big) \mathrm{d}\theta\,; \quad \theta \in \Delta_n^+\,.$$

- Now we shall see the universal portfolio is generated by a function in the sense of FERNHOLZ ('02).

Let $F : \Delta_+^n \times \mathbb{R}^m \to \mathbb{R}$ be a smooth function that is twice continuously differentiable in the first n variables in Δ_+^n, continuously differentiable in the last m variables in $\mathbb{R}^m$.

We denote by D_i the derivative with respect to i-th variable and $D_{i,j} := D_i D_j$. We shall impose that $x_i D_i \log F(x, y)$ is bounded in $\Delta_+^n \times \mathbb{R}^m$ for $i = 1, \ldots, n$, that is,

$$\max_{1 \le i \le n} \sup_{(x,y) \in \Delta_+^n \times \mathbb{R}^m} \left| x_i D_i \log F(x, y) \right| < \infty .$$

one can consider a portfolio $\pi^{F,\phi}(\cdot)$ generated by $F(\cdot)$, $\mu(\cdot)$ and some $\mathbb{G}$-progressively measurable, continuous process $\phi(\cdot) := (\phi_1(\cdot), \ldots, \phi_m(\cdot))$ of finite variation :

$$\pi_i^{F,\phi}(t) := \Big[D_i \log F(\mu(t), \phi(t)) + 1 - \sum_{j=1}^{n} \mu_j(t) D_j \log F(\mu(t), \phi(t)) \Big] \mu_i(t)$$

for $i = 1, \ldots, n$, $0 \le t < \infty$.

Proposition 1 The wealth process $V^{v,\pi^{F,\phi}}(\cdot)$ of the portfolio $\pi^{F,\phi}(\cdot)$ generated by the above set $(F(\cdot), \phi(\cdot))$ of function and finite-variation process is compared with the market portfolio wealth process $V^{v,\mu}(\cdot)$ by

$$\mathrm{d}\log\left(\frac{V^{v,\pi^{F,\phi}}(t)}{V^{v,\mu}(t)}\right) = \mathrm{d}\log F(\mu(t), \phi(t)) + \mathrm{d}\Theta(t),$$

$$\mathrm{d}\Theta(t) := \frac{(-1)}{2F(\mu(t),\phi(t))} \sum_{i,j=1}^{n} D_{i,j} F(\mu(t),\phi(t))\, \mu_i(t)\, \mu_j(t)\, \alpha_{i,j}(t)\, \mathrm{d}t - \sum_{\ell=1}^{m} D_{\ell+n} \log F(\mu(t),\phi(t))\, \mathrm{d}\phi_\ell(t)$$

for $0 \le t < \infty$, where $(\alpha_{i,j}(\cdot))_{1 \le i,j \le n} = \sigma(\cdot)\sigma'(\cdot)$.

Proof.
Follow the proof of Fernholz ('02). See also Strong ('12) and Brod ('14). □

Given the initial market weights $\mu(0) := (\mu_1(0), \ldots, \mu_n(0))$, let us apply Proposition 1 with function

$$F(x, y) = F(x_1, \ldots, x_n, y_1, \ldots, y_m)$$

$$:= \int_{\Delta_+^n} \Big[\prod_{i=1}^n \Big(\frac{x_i}{\mu_i(0)} \Big)^{\theta_i} \Big] \cdot \exp \Big[\frac{1}{2} \Big(\sum_{j=1}^n \theta_j \, y_{c(j,j)} - \sum_{i,j=1}^n \theta_i \, \theta_j \, y_{c(i,j)} \Big) \Big] \mathrm{d}\theta$$

and the $\mathbb{G}$-progressively measurable process $\phi(\cdot) = (\phi_1(\cdot), \ldots, \phi_m(\cdot))'$ with

$$\phi_{c(i,j)}(\cdot) := \int_0^{\cdot} \alpha_{i,j}(s) \mathrm{d}s$$

of finite variation, where the indices $c(i,j)$ are defined by

$$c(i,j) := \begin{cases} i & \text{if} \quad i = j\,, \\ j + [\; i(2n - i - 1) / 2 \;] & \text{if} \quad i < j\,, \\ i + [\; j(2n - j - 1) / 2 \;] & \text{if} \quad i > j \end{cases}$$

for $1 \le i, j \le n$ with $m := n(n+1)/2$. Here $c(1,2) = n+1, \ldots, c(1,n) = 2n-1, \ldots, c(n-1,n) = m$.

Note that by direct calculations

$$\sum_{i=1}^{n} \mu_i(t) D_i \log F(\mu(t), \phi(t)) = 1 .$$

Thus we observe the resulting portfolio

$$\begin{aligned}\pi_i^{F,\phi}(t) &= \Big[D_i \log F(\mu(t), \phi(t)) + 1 - \sum_{j=1}^{n} D_j \log F(\mu(t), \phi(t))\Big] \mu_i(t) \\ &= \mu_i(t) D_i \log F(\mu(t), \phi(t)) \\ &= \big[F(\mu(t), \phi(t))\big]^{-1} \Big[\int_{\Delta_+^n} \theta_i \cdot \prod_{j=1}^{n} \Big(\frac{\mu_j(t)}{\mu_j(0)}\Big)^{\theta_j} \cdot \exp\big[\varphi^*(\theta, t)\big] \mathrm{d}\theta\Big]\end{aligned}$$

and then

$$\pi_i^{F,\phi}(t) = \int_{\Delta_+^n} \theta_i \, \mathrm{d}\nu^{**}(\mathrm{d}\theta, \cdot) = \pi_i^{u}(t) , \quad i = 1, \ldots, n , \quad t \geq 0 ,$$

where $\nu^{**}(\mathrm{d}\theta, \cdot)$ is the measure-valued process.

Proposition 2 The universal portfolio $\pi^u(\cdot)$ is functionally generated by (F, ϕ), i.e., $\pi^u(\cdot) = \pi^{F,\phi}(\cdot)$. Its wealth process $V^{v,\pi^u}(\cdot)$ is compared with the market portfolio wealth process $V^{v,\mu}(\cdot)$ by

$$\frac{V^{v,\pi^{F,\phi}}(t)}{V^{v,\mu}(t)} = \frac{F(\mu(t), \phi(t))}{F(\mu(0), \phi(0))}$$

for $t \geq 0$.

Here there is no finite variation part in the comparison formula, i.e.,

$$\Theta(\cdot) \equiv 0$$

in the comparison of Proposition 1:

$$\mathrm{d} \log \left(\frac{V^{v,\pi^{F,\phi}}(\cdot)}{V^{v,\mu}(\cdot)} \right) = \mathrm{d} \log F(\mu(\cdot), \phi(t)) + \mathrm{d}\Theta(\cdot) .$$

• This answers a question by Fernholz & Karatzas ('09) : Connection between portfolio generating functions and universal portfolios ? Non trivial example of $\Theta(\cdot) \equiv 0$?

• Pal & Wong ('14) : for portfolios generated by positive concave function $\Phi(\cdot)$ of n variables $(\mu_1(\cdot), \ldots, \mu_n(\cdot))$, finite variation part of $\log(V^{v,\pi}(\cdot) / V^{v,\mu}(\cdot))$ is zero (i.e., $\Theta(\cdot) = 0$),

if and only if $\Phi(\cdot)$ is affine.

Here $\Theta(\cdot) \equiv 0$, although $F(\cdot, \phi(t))$ is not affine.

• The measure $\nu^*(\cdot, t)$ is the solution to the problem of maximizing the entropy

$$\nu^*(\cdot, t) = \arg \max_{\nu \in S_{a(t)}} \Big[- \int_{\Delta_+^n} \log \Big(\frac{\mathrm{d}\nu}{\mathrm{d}m}(\theta) \Big) \nu(\mathrm{d}\theta) \Big],$$

among $\nu(\cdot) \in S_{a(t)}$, where $m(\cdot)$ is the uniform probability measure on Δ_+^n and $S_{a(t)}$ is the family

$$S_{a(t)} := \Big\{ \nu(\ll m) : \int_{\Delta_+^n} \varphi^*(\theta, t) \, \nu(\mathrm{d}\theta) = a(t) \Big\}$$

for some $\mathbb{G}$-progressively measurable process $a(\cdot)$.

Corollary: Arbitrage Relative to Market

Let us define a performance measure of the market :

$$\mathcal{S}(t) := \sum_{i=1}^{n} \log\left(\frac{\mu_i(t)}{\mu_i(0)}\right) \int_{\Delta_+^n} \theta_i \, \nu^*(\mathrm{d}\theta; t) ; \quad t \geq 0 .$$

Then we have almost surely

$$\frac{V^{v,\pi^u}(t)}{V^{v,\mu}(t)} \geq \exp\left(\mathcal{S}(t) + \log\left(\int_{\Delta_+^n} e^{\varphi^*(\theta,T)} \mathrm{d}\theta \Big/ \int_{\Delta_+^n} \mathrm{d}\theta\right)\right) ; \quad t \geq 0 .$$

In particular, if

$$\mathbb{P}\Big(\mathcal{S}(T) + \log\left(\int_{\Delta_+^n} e^{\varphi^*(\theta,T)} \mathrm{d}\theta \Big/ \int_{\Delta_+^n} \mathrm{d}\theta\right) \geq 0\Big) = 1 ,$$

$$\mathbb{P}(\mathcal{S}(T) + \log\left(\int_{\Delta_+^n} e^{\varphi^*(\theta,T)} \mathrm{d}\theta \Big/ \int_{\Delta_+^n} \mathrm{d}\theta\right) > 0) > 0$$

for some $T > 0$, then the universal portfolio is an arbitrage opportunity relative to the market, i.e.,

$$\mathbb{P}(V^{v,\pi^u}(T) \geq V^{v,\mu}(T)) = 1 , \quad \mathbb{P}(V^{v,\pi^u}(T) > V^{v,\mu}(T)) > 0 .$$

Example: Two stocks

$$\gamma_\theta^*(\cdot) = \frac{1}{2}\big(\theta\alpha_{1,1}(\cdot) + (1-\theta)\alpha_{2,2}(\cdot) - \theta^2\alpha_{1,1}(\cdot) + 2\theta(1-\theta)\alpha_{1,2}(\cdot) + (1-\theta)^2\alpha_{2,2}(\cdot)\big) = -\frac{\alpha^\circ(\cdot)}{2}(\theta^2 - \theta)\,,$$

where $\alpha^\circ(\cdot) := \alpha_{1,1}(\cdot) + \alpha_{2,2}(\cdot) - 2\alpha_{1,2}(\cdot)$.
Let us write $\alpha^\bullet(\cdot) := \int_0^\cdot \alpha^\circ(s)\mathrm{d}s$, $\mathfrak{f}(\cdot) := (\alpha^\bullet(\cdot))^{1/2}$ and $\mu_i^\circ(\cdot) := \mu_i(\cdot)\,/\,\mu_i(0)$ for $i = 1, 2$ for simplicity. Then

$$\varphi^*(\theta, \cdot) = -(1/2)\alpha^\bullet(\cdot)(\theta^2 - \theta)\,.$$

The relative performance of universal portfolio $\pi^u(\cdot)$ to the market portfolio $\mu(\cdot)$ is

$$\frac{V^{v,\pi^u}(\cdot)}{V^{v,\mu}(\cdot)} = \int_0^1 \Big[\frac{\mu_1(\cdot)}{\mu_1(0)}\Big]^\theta \cdot \Big[\frac{\mu_2(\cdot)}{\mu_2(0)}\Big]^{1-\theta} \cdot e^{\varphi^*(\theta,\cdot)}\mathrm{d}\theta\,.$$

$$e^{-(1/8)\alpha^\bullet(\cdot)} \frac{V^{v,\pi^u}(\cdot)}{V^{v,\mu}(\cdot)} = \int_0^1 \left[\frac{\mu_1(\cdot)}{\mu_1(0)}\right]^\theta \cdot \left[\frac{\mu_2(\cdot)}{\mu_2(0)}\right]^{1-\theta} \cdot e^{\varphi^*(\theta,\cdot)} \mathrm{d}\theta \cdot e^{-(1/8)\alpha^\bullet(\cdot)}$$

$$= \left(\int_0^{1/2} + \int_{1/2}^1\right)\left\{\left[\mu_1^\circ(\cdot)\right]^\theta \cdot \left[\mu_2^\circ(\cdot)\right]^{1-\theta} \cdot e^{-(1/2)\alpha^\bullet(\cdot)(\theta-(1/2))^2}\right\}\mathrm{d}\theta$$

$$= \int_0^{1/2} \left(\left[\mu_1^\circ(\cdot)\right]^\theta \cdot \left[\mu_2^\circ(\cdot)\right]^{1-\theta} + \left[\mu_1^\circ(\cdot)\right]^{1-\theta} \cdot \left[\mu_2^\circ(\cdot)\right]^\theta\right) e^{-(1/2)\alpha^\bullet(\cdot)(\theta-1/2)^2} \mathrm{d}\theta$$

and then because of *inequality of arithmetic and geometric means*, we may have the lower bound

$$\frac{V^{v,\pi^u}(\cdot)}{V^{v,\mu}(\cdot)} = \frac{e^{\frac{\alpha^\bullet(\cdot)}{8}}}{\mathfrak{f}(\cdot)} \int_0^{\frac{\mathfrak{f}(\cdot)}{2}} \left[\mu_1^\circ(\cdot) \cdot \mu_2^\circ(\cdot)\right]^{\frac{1}{2}} \left[\left(\frac{\mu_2^\circ(\cdot)}{\mu_1^\circ(\cdot)}\right)^{\frac{u}{\mathfrak{f}(\cdot)}} + \left(\frac{\mu_1^\circ(\cdot)}{\mu_2^\circ(\cdot)}\right)^{\frac{u}{\mathfrak{f}(\cdot)}}\right] e^{-\frac{u^2}{2}} \mathrm{d}u$$

$$\geq \left[\mu_1^\circ(\cdot) \cdot \mu_2^\circ(\cdot)\right]^{1/2} \cdot e^{(\mathfrak{f}(\cdot))^2/8} \cdot \frac{1}{\mathfrak{f}(\cdot)} \int_0^{\mathfrak{f}(\cdot)/2} e^{-u^2/2} \mathrm{d}u$$

$$= \left[\frac{\mu_1(\cdot)}{\mu_1(0)} \cdot \frac{\mu_2(\cdot)}{\mu_2(0)}\right]^{1/2} \cdot \mathfrak{g}(\mathfrak{f}(\cdot)),$$

where

$$\mathfrak{g}(c) := e^{c^2/8} \cdot \frac{1}{c} \int_0^{c/2} e^{-u^2/2} \mathrm{d}u ; \quad c \geq 0$$

is an increasing function in $c \geq 0$ with $\mathfrak{g}(0) = 1/2$, and has its inverse function $\mathfrak{g}^{-1}(\cdot)$ in $\mathbb{R}_+$. Thus sufficient volatility creates relative arbitrage to the market, i.e., for every $\varepsilon \geq 0$

$$V^{v,\pi^u}(\tau_\varepsilon) \geq (1+\varepsilon) V^\mu(\tau_\varepsilon),$$

where τ_ε is the first time that cumulative volatility is large enough, i.e.,

$$\tau_\varepsilon := \inf \Big\{ t : \Big(\int_0^t (\alpha_{1,1}(s) + \alpha_{2,2}(s) - 2\alpha_{1,2}(s)) \mathrm{d}s \Big)^{1/2} \geq \mathfrak{g}^{-1} \Big((1+\varepsilon) \Big[\frac{\mu_1(\cdot)}{\mu_1(0)} \cdot \frac{\mu_2(\cdot)}{\mu_2(0)} \Big]^{-1/2} \Big) \Big\}.$$

In particular, if the market model is diverse (see section 2.2 of FERNHOLZ (2002)), i.e., if there exists $\delta \in (0, 1/2)$ such that $\max(\mu_1(\cdot), \mu_2(\cdot)) \leq 1 - \delta$, and if the market is not degenerate in the sense of

$$\alpha_{1,1}(\cdot) + \alpha_{2,2}(\cdot) - 2\alpha_{1,2}(\cdot) \geq \eta^2$$

for some $\eta > 0$, then the universal portfolio is a strong arbitrage relative to the market :

$$\mathbb{P}(V^{v,\pi^u}(T) \geq (1+\varepsilon) V^{v,\mu}(T)) = 1$$

for every

$$T \geq \frac{1}{\eta^2} \cdot \Big[\mathfrak{g}^{-1}\Big((1+\varepsilon)\Big(\frac{1-\delta}{\delta}\Big)\Big)\Big]^2 .$$

Thus the conclusion of the corollary holds in a stronger form.
□

- Diversity : FERNHOLZ, KARATZAS, KARDARAS ('05), ...
- Optimal : FERNHOLZ & KARATZAS ('08), ...

A case study: comparisons under Atlas model

Universal Portfolio versus Portfolio $\pi^G(\cdot)$ generated by $(G(\cdot), \mu(\cdot))$,

$$\pi^G(t) := \Big[D_i \log G(\mu(t)) + 1 - \sum_{j=1}^{n} \mu_j(t) D_j \log G(\mu(t)) \Big] \mu_i(t)$$

for $i = 1, \ldots, n$ with wealth process $V^{v,\pi^G}(t)$ $0 \le t < \infty$.
Let us consider more specific dynamics of log stock price
$Y_i(\cdot) := \log(X_i(\cdot))$ with reverse order statistics:
$Y_{(1)}(\cdot) \ge Y_{(2)}(\cdot) \ge \cdots \ge Y_{(n)}(\cdot)$:

$$\mathrm{d}\, Y_i(t) = \Big(\sum_{k=1}^{n} \delta_k \cdot \mathbf{1}_{\{ Y_i(t) = Y_{(k)}(t) \}} \Big) \mathrm{d}t + \mathrm{d}\, W_i(t)$$

for $i = 1, \ldots, n$, $t \ge 0$ for simplicity.

- Atlas model (Banner, Fernholz & Karatzas ('05))

Under the stability condition $\delta_1 + \cdots + \delta_n = 0$ and

$$\sum_{\ell=1}^{k} \delta_{\mathbf{p}(\ell)} < 0 \,, \quad k = 1, \ldots, n-1$$

for every $\mathbf{p}$ of symmetric group of permutations of $\{1, \ldots, n\}$, the gaps $(Y_{(1)}(\cdot) - Y_{(2)}(\cdot), \ldots, Y_{(n-1)}(\cdot) - Y_{(n)}(\cdot))$ have joint exponential distribution $\boldsymbol{\kappa}(\cdot)$ as unique stationary distribution.

Proposition 3 Under some additional assumptions by *concentration of measure phenomenon* we have

$$\mathbb{P}_{\boldsymbol{\nu}}\Big(\frac{V^{\upsilon, \pi^u}(t)}{V^{\upsilon, \pi^G}(t)} \ge e^{(r - \boldsymbol{\nu}(\widetilde{u}))t} \cdot \frac{G(\mu(0))F(\mu(t), \phi(t))}{G(\mu(t))F(\mu(0), \phi(0))} \Big)$$

$$\le \Big\| \frac{\mathrm{d}\boldsymbol{\kappa}}{\mathrm{d}\boldsymbol{\nu}} \Big\|_{L^2(\boldsymbol{\nu})} \cdot \exp\big(- c\beta t \big) \,,$$

for every r, t, where

$$c := \max\Big(\frac{r^2}{\delta^2(u)}, 4\varepsilon(\varepsilon+\sigma^2)\sqrt{1+\frac{r^2}{2\varepsilon(\varepsilon+\sigma^2)^2\|u\|_\infty^2}} - 1\Big),$$

$$\beta := 2\big(\min_{1\le k\le n-1}\big[\sum_{\ell=1}^{k}\delta_\ell\big]^2\big)(1-\cos(\pi / n)),$$

$$\sigma^2 := \int_{(\mathbb{R}_+)^{n-1}} u^2(y)\boldsymbol{\nu}(\mathrm{d}y) - \Big(\int_{(\mathbb{R}_+)^{n-1}} u(y)\boldsymbol{\nu}(\mathrm{d}y)\Big)^2,$$

$$\delta(u) := \sup|u(x)-u(y)|, \quad u(y) := \widetilde{u}(y) - \int_{(\mathbb{R}_+)^{n-1}} \widetilde{u}(y)\boldsymbol{\nu}(\mathrm{d}y),$$

and $\widetilde{u}(\cdot)$ is chosen so that

$$\widetilde{u}(Y_{(1)}(\cdot),\dots,Y_{(n)}(\cdot)) = \frac{(-1)}{2G(\mu(t))}\sum_{i,j=1^n} D_{i,j}G(\mu(t))\frac{\mathrm{d}\langle\mu_i,\mu_j\rangle}{\mathrm{d}t}(t)$$

for every $t \ge 0$ (cf. Ichiba, Shkolnikov & Pal ('13)).

Summary.

- Properties of universal portfolios
- Connection to portfolios generated by functions
- Concentration of measure inequality
  - Large equity market model

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