Novikov-type conditions for processes with jumps

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- **Introduction**
- **Two tools:** The Föllmer measure and reciprocals of stochastic exponentials
- **Some history:** Novikov ('72), Kazamaki ('77), Lépingle & Mémin ('78), Kallsen & Shiryaev ('02)
- **Two general criteria**

Novikov's and Kazamaki's criteria

- $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$ stochastic basis, $Z > 0$ continuous local martingale
- Then there exists a continuous local martingale M such that

$$Z = \mathscr{E}(M) := \exp\left\{M - \frac{1}{2}\langle M, M\rangle\right\}$$

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Question: Is $\mathscr{E}(M)$ a u.i. martingale? Or, $\exists \mathbb{Q} \sim \mathbb{P}$ with $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \mathscr{E}(M)_\infty$?

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Theorem (Novikov '72).

$$\mathbb{E}\left[e^{\frac{1}{2}\langle M,M\rangle_\infty}\right] < \infty \qquad \Longrightarrow \qquad \mathscr{E}(M) \text{ is a u.i. martingale.}$$

Theorem (Kazamaki '77). Assume M is a u.i. martingale. Then

$$\mathbb{E}\left[e^{\frac{1}{2}M_\infty}\right] < \infty \qquad \Longrightarrow \qquad \mathscr{E}(M) \text{ is a u.i. martingale.}$$

Extension to the jump case

Setup:

- $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$ stochastic basis, $\mathbb{F}$ right-continuous, not necessarily $\mathbb{P}$-augmented
- $\mathbb{F}$ is right-continuous modification of a standard system (w.l.o.g.)
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- $Z \geq 0$ local martingale, $\tau_0 = \inf\{t : Z_t = 0\}$
- Assume Z does not jump to zero: $Z_{\tau_0 -} = 0$ on $\{\tau_0 < \infty\}$. Then:
  - τ_0 is foretellable:

$$\tau_n := \inf\{t : Z_t \leq n^{-1}\} \ < \ \tau_0, \qquad \lim_{n\to\infty} \tau_n = \tau_0 \qquad \mathbb{P}\text{-a.s.}$$

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  - There exists a local martingale M on $[\![0, \tau_0[\![$ with $Z = \mathscr{E}(M)$,

$$\mathscr{E}(M) := \exp\left\{M - \frac{1}{2}\langle M^c, M^c\rangle - (x - \log(1+x)) * \mu^M\right\} \mathbf{1}_{[\![0,\tau_0[\![}$$

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- Notation: For càdlàg adapted X, μ^X is its jump measure, ν^X the predictable compensator.

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- ... as well as generalizations, including necessary and sufficient conditions
- Key idea: change measure without uniform integrability, then apply non-explosion criteria

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Theorem (∗) (Föllmer '72, Meyer '72, etc.). There exists a probability measure $\mathbb{Q}$ on $\mathscr{F}$, unique on $\mathscr{F}_{\tau_\infty -}$, such that

$$\mathbb{E}_{\mathbb{Q}}\left[X\,\mathbf{1}_{\{\sigma<\tau_\infty\}}\right] = \mathbb{E}_{\mathbb{P}}\left[X\,Z_\sigma\right]$$

for any bounded stopping time σ and $\mathscr{F}_\sigma$-measurable $X \geq 0$. Moreover,

$$\begin{aligned}\mathbb{P} \sim \mathbb{Q} \quad &\Longleftrightarrow \quad Z \text{ is a } \mathbb{P}\text{-u.i. martingale} \\ &\Longleftrightarrow \quad \mathbb{Q}\Big(\sup_{t<\tau_\infty} Z_t < \infty\Big) = 1 \text{ (in particular, } \tau_\infty = \infty\ \mathbb{Q}\text{-a.s.)}\end{aligned}$$

Note: $\dfrac{1}{Z}\mathbf{1}_{[\![0,\tau_\infty[\![}$ is a nonnegative local $\mathbb{Q}$-martingale

Proof of Novikov's criterion

Theorem (Novikov '72). Assume M is continuous and $\tau_0 = \infty$ $\mathbb{P}$-a.s. Then

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Proof:

- Write
$$\frac{1}{Z} = e^{-M+\langle M,M\rangle - \frac{1}{2}\langle M,M\rangle} = \mathscr{E}(N),$$
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- Novikov's criterion forces $\limsup_{t\uparrow\tau_\infty} N_t < \infty$ $\mathbb{Q}$-a.s.
- Hence $\tau_\infty = \infty$ and $\mathscr{E}(N)_\infty > 0$ $\mathbb{Q}$-a.s., so we are done by Theorem $(*)$. $\square$

The reciprocal of a stochastic exponential

- A key step in the proof of Novikov's criterion was to write $1/Z = \mathscr{E}(N)$ for some local $\mathbb{Q}$-martingale N on $[\![0, \tau_\infty[\![$.

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Lemma. Define a semimartingale N on $[\![0, \tau_0[\![$ by

$$N = -M + \langle M^c, M^c \rangle + \frac{x^2}{1+x} * \mu^M.$$

Then

$$\mathscr{E}(M)\mathscr{E}(N) = 1 \quad \text{on} \quad [\![0, \tau_0[\![.$$

Also, $\Delta N = \phi(\Delta M)$. Hence for any predictable function $W \geq 0$,

$$W * \mu^M = (W \circ \phi) * \mu^N$$

where $(W \circ \phi)(\omega, t, x) := W(\omega, t, \phi(x))$. The same formula holds with ν^M, ν^N instead of μ^M, μ^N.

Two criteria by Lépingle and Mémin

Theorem (Lépingle & Mémin '78a). Assume $\tau_0 = \infty$ $\mathbb{P}$-a.s. Define

$$A = \frac{1}{2}\langle M^c, M^c\rangle + \left(\log(1+x) - \frac{x}{1+x}\right) * \mu^M$$

$$B = \frac{1}{2}\langle M^c, M^c\rangle + ((1+x)\log(1+x) - x) * \nu^M.$$

If either $\mathbb{E}\left[e^{A_\infty}\right] < \infty$ or $\mathbb{E}\left[e^{B_\infty}\right] < \infty$ holds, then $Z = \mathscr{E}(M)$ is a u.i. martingale.

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Proof (criterion involving A):

$$\begin{aligned} A &= \frac{1}{2}\langle M^c, M^c\rangle + \left(\log(1+x) - \frac{x}{1+x}\right) * \mu^M \\ &= \left[M - \frac{1}{2}\langle M^c, M^c\rangle - (x - \log(1+x)) * \mu^M\right] + \left[-M + \langle M^c, M^c\rangle + \frac{x^2}{1+x} * \mu^M\right] \\ &= \log Z + N. \end{aligned}$$

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Hence $\mathbb{E}\left[e^{A_\sigma}\right] = \mathbb{E}_{\mathbb{Q}}\left[e^{N_\sigma}\mathbf{1}_{\{\sigma<\tau_\infty\}}\right]$.

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Lemma. Let X be a local martingale on a stochastic interval $[\![0,\tau[\![$. If

$$\sup_\sigma \mathbb{E}\left[e^{X_\sigma}\mathbf{1}_{\{\sigma<\tau\}}\right] < \infty$$

then $\lim_{t\uparrow\tau} X_t$ exists in $\mathbb{R}$, almost surely.

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Hence N converges, so $\tau_\infty = \infty$ and $\mathscr{E}(N)_\infty > 0$ $\mathbb{Q}$-a.s. □

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Proof (criterion involving B):

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Lemma. For any predictable function $W \geq 0$ and any bounded stopping time $\sigma < \tau$ such that $Z^\sigma = \mathscr{E}(M)^\sigma$ is a martingale,

$$\mathbb{E}\left[\mathscr{E}(M)_\sigma \, (W * \mu^N)_\sigma\right] = \mathbb{E}\left[\mathscr{E}(M)_\sigma \, (W * \frac{\nu^N_\sigma}{1+y})_\sigma\right].$$

In particular, $\nu^{N,\mathbb{Q}} := (1+y)^{-1}\nu^N$ is the compensator of μ^N under $\mathbb{Q}$.

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Hence $B = \frac{1}{2}\langle M^c, M^c\rangle + (y - \log(1+y)) * \nu^{N,\mathbb{Q}}$.

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- Thus, on $[\![0, \tau_\infty[\![$,

$$\mathscr{E}(N) = e^{N^c + \log(1+y)*(\mu^N - \nu^{N,\mathbb{Q}}) - B}$$

or equivalently,

$$e^B = \mathscr{E}(M) e^{N^c + \log(1+y)*(\mu^N - \nu^{N,\mathbb{Q}})}$$

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$$\sup_\sigma \mathbb{E}_\mathbb{Q}\left[e^{N^c_\sigma + \log(1+y)*(\mu^N - \nu^{N,\mathbb{Q}})_\sigma} \mathbf{1}_{\{\sigma < \tau_\infty\}}\right] < \infty \qquad (*)$$

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Theorem. Condition $(*)$ implies that N converges $\mathbb{Q}$-a.s.

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Theorem. Condition $(*)$ implies that N converges $\mathbb{Q}$-a.s.

Consequently, $\tau_\infty = \infty$ and $\mathscr{E}(N)_\infty > 0$ $\mathbb{Q}$-a.s. □

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- If this is even a martingale, then $e^{K^X(\theta)_t} = \mathbb{E}[e^{\theta \cdot X_t} \mid \mathscr{F}_{t-}]$. Hence $K^X(\theta)$ is a "conditional cumulant generating function".

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- If this is even a martingale, then $e^{K^X(\theta)_t} = \mathbb{E}[e^{\theta \cdot X_t} \mid \mathscr{F}_{t-}]$. Hence $K^X(\theta)$ is a "conditional cumulant generating function".
- The **derivative process** $DK^X(\theta)$ can always be defined. It can sometimes be interpreted as "$\frac{d}{d\theta} K^X(\theta)$".

A criterion by Kallsen and Shiryaev

- X a semimartingale (one-dimensional for simplicity)
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Theorem (Kallsen & Shiryaev '02). Assume X is quasi-left continuous and $|xe^x - x\mathbf{1}_{\{|x| \leq 1\}}| * \nu^X$ is a finite-valued process. If

$$\sup_\sigma \mathbb{E}\left[e^{DK^X(1)_\sigma - K^X(1)_\sigma}\right] < \infty$$

then $Z = e^{X - K^X(1)}$ is a u.i. martingale.

A criterion by Kallsen and Shiryaev

Proof:

- Let M be a local martingale with $\Delta M > 1$ such that

$$Z = e^{X - \kappa^X(1)} = \mathscr{E}(M)$$

- Define N as before: $\mathscr{E}(M)\mathscr{E}(N) = 1$

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Hence on $[\![0, \tau_\infty[\![$,

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so that

$$\mathbb{E}\left[e^{DK^X(1)_\sigma - K^X(1)_\sigma}\right] = \mathbb{E}_{\mathbb{Q}}\left[e^{N^c_\sigma + \log(1+y)*(\mu^N - \nu^{N,\mathbb{Q}})_\sigma} \mathbf{1}_{\{\sigma < \tau_\infty\}}\right]$$

The result now follows as before. $\square$

Two general criteria

- The following concept plays a key role for convergence under $\mathbb{Q}$:

  A process U on $[\![0, \tau_\infty[\![$ is extended locally integrable (under $\mathbb{Q}$) if there exist stopping times ρ_m such that

$$\lim_{m\to\infty} \mathbb{Q}(\rho_m = \tau_\infty) = 1 \quad \text{and} \quad \sup_{t<\tau_\infty} |U_t^{\rho_m}| \in L^1(\mathbb{Q}) \quad \forall\, m$$

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- How can this property be characterized under $\mathbb{P}$?
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Lemma. Let U be a nondecreasing càdlàg adapted process on $[\![0, \tau_\infty[\![$, null at zero. Then U is extended locally integrable under $\mathbb{Q}$ if and only if there exists a nondecreasing sequence (ρ_m) of stopping times such that

(i) $\lim_{m\to\infty} \mathbb{E}_\mathbb{P}[\mathbf{1}_{\{\rho_m<\infty\}} Z_{\rho_m}] = 0$

(ii) $\sup_n \mathbb{E}_\mathbb{P}[U_{\tau_n^+\wedge\rho_m} Z_{\tau_n^+\wedge\rho_m}] < \infty$ for all m.

The set of U satisfying (i) and (ii) is denoted $\mathscr{A}^+_{\mathrm{ext},Z}$

Two general criteria

Theorem. For any $a \in \mathbb{R}$, define on $[\![0, \tau_0[\![$ a process

$$A^a = aM + \left(\frac{1}{2} - a\right)\langle M^c, M^c\rangle + \left(\log(1+x) - \frac{ax^2 + x}{1+x}\right) * \mu^M$$

The following are equivalent:

(i) $Z = \mathscr{E}(M)$ is a u.i. martingale

(ii) For some $a \neq 1$ and some $U \in \mathscr{A}^+_{\mathrm{ext},Z}$,

$$\sup_\sigma \mathbb{E}\left[e^{A^a_\sigma - U_\sigma}\mathbf{1}_{\{\sigma<\tau_0\}}\right] < \infty$$

- With $U = 0$ we improve results by Lépingle & Mémin ('78b)

Two general criteria

Theorem. For any $a \in \mathbb{R}$, define on $[\![0, \tau_0[\![$,

$$B^a = aM + \left(\frac{1}{2} - a\right)\langle M^c, M^c\rangle - a(x - \log(1+x)) * \mu^M + (1-a)((1+x)\log(1+x) - x) * \nu^M$$

The following are equivalent:

(i) $Z = \mathscr{E}(M)$ is a u.i. martingale and for some constant $\kappa > 0$, $((1+x)\log(1+x)\mathbf{1}_{x>\kappa} * \nu^M \in \mathscr{A}^+_{\mathrm{ext},Z}$

(ii) $(1+x)^{-1}\mathbf{1}_{x<-\varepsilon} * \nu^M \in \mathscr{A}^+_{\mathrm{ext},Z}$ for some $\varepsilon \in (0,1)$, and for some $a \neq 1$ and some $U \in \mathscr{A}^+_{\mathrm{ext},Z}$,

$$\sup_\sigma \mathbb{E}\left[e^{B^a_\sigma - U_\sigma}\mathbf{1}_{\{\sigma<\tau_0\}}\right] < \infty$$

Conclusion

- Transparent proofs of many known Novikov-Kazamaki type conditions
- Idea: Combine the Föllmer measure with non-explosion criteria for local martingales
- Extensions of many known criteria ...
- ... including necessary and sufficient conditions