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Mathematical Finance: Arbitrage and Portfolio Optimization

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# Arbitrages and progressive enlargement of filtration

Based on works with Anna Aksamit, Tahir Choulli, Jun Deng.

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LaMME Laboratoire de Mathématiques et Modélisation d'Évry

L’avenir est la projection du passé, conditionnée par le présent. Georges Braque.

The future is the projection of the past, conditionally to the present. Georges Braque.

The goal of this presentation if to analyze wether or not more information gives arbitrage opportunities in a financial market.

We shall present the case of progressive enlargement and we study NUPBR.

# Enlargement of filtration results

As in Beatrix's talk, we define the $\mathbb{F}$-supermartingale

$$Z_t := \mathbb{P}\left(\tau > t \,\middle|\, \mathcal{F}_t\right).$$

Denoting by $A^0$ is the dual optional projection of $1\!\!1_{\tau \leq t}$, the process $m = Z + A^0$ is an $\mathbb{F}$ martingale.

In the special case where $\tau$ **avoids** $\mathbb{F}$ **stopping times**, we can write the Doob-Meyer decomposition of $Z$ as

$$Z = m - A^p = m - A^o$$

where $A^0 = A^p$ is a (predictable) continuous increasing process

# Before $\tau$

From Jeulin, for any $\mathbb{F}$ local martingale $X$ the process

$$X_t^{\mathbb{G}} := X_t^{\tau} - \int_0^{t\wedge\tau} \frac{1}{Z_{s-}} d\langle X, m\rangle_s^{\mathbb{F}}$$

is a $\mathbb{G}$ local martingale (stopped at time $\tau$).

In the case where all $\mathbb{F}$ martingales are continuous and $\tau$ avoids $\mathbb{F}$ stopping times, it is easy to see that **NA of the first kind holds before $\tau$.**

Indeed, Itô's calculus shows that $d\ell_t = -\ell_t \kappa_t dm^{\mathbb{G}}_t$ is a deflator for $\kappa = \frac{1}{Z}$

The following results holds true (Aksamit et al., See also Acciaio et al. )

**For any (bounded) $X$ satisfying NUPBR$(\mathbb{F})$, $X^\tau$ satisfies NUPBR$(\mathbb{G})$ if and only if the thin set $\left\{\widetilde{Z} = 0 \;\&\; Z_- > 0\right\}$ is evanescent.**

Here, $\widetilde{Z}_t = \mathbb{P}(\tau \geq t|\mathcal{F}_t)$.

Our goal now is to explain in a naive way why that "evanescent " condition is needed.

If $\mathbb{F}$ is the filtration generated by a Poisson process $N$, with compensated martingale $M$, we are looking for a $\mathbb{G}$-local martingale deflator of the form $d\ell_t = \ell_{t-}\kappa_t dm_t^{\mathbb{G}}$ so that $\ell$ is positive and $S^\tau \ell$ is a $\mathbb{G}$-local martingale, where $dS_t = S_{t-}\psi_t dM_t$.

From PRT in the filtration $\mathbb{F}$, $dm_t = \nu_t dM_t$. Integration by parts formula leads to (on $t \leq \tau$).

$$\begin{aligned}
d(\ell S)_t &= \ell_{t-}dS_t + S_{t-}d\ell_t + d[\ell, S]_t \\
&\overset{\mathbb{G}-\text{mart}}{=} \ell_{t-}S_{t-}\psi_t \frac{1}{Z_{t-}} d\langle M, m\rangle_t + \ell_{t-}S_{t-}\kappa_t\psi_t\nu_t dN_t \\
&\overset{\mathbb{G}-\text{mart}}{=} \ell_{t-}S_{t-}\psi_t \frac{1}{Z_{t-}} \nu_t\lambda dt + \ell_{t-}S_{t-}\kappa_t\psi_t\nu_t\lambda(1 + \frac{1}{Z_{t-}}\nu_t)dt \\
&= \ell_{t-}S_{t-}\psi_t\nu_t\lambda \left( \frac{1}{Z_{t-}} + \kappa_t(1 + \frac{1}{Z_{t-}}\nu_t) \right) dt.
\end{aligned}$$

Therefore, for $\kappa_t = -\frac{1}{Z_{t-}+\nu_t}$, one obtains (formally) a deflator

$$d\ell_t = -\ell_{t-}\frac{\nu_t}{Z_{t-} + \nu_t} dM_t^{\mathbb{G}}$$

One needs that the quantity

$$d\ell_t = -\ell_{t-} \frac{\nu_t}{Z_{t-} + \nu_t} dM_t^{\mathbb{G}}$$

is well defined. Since, at jump times of the Poisson process

$$Z_{t-} + \nu_t = Z_{t-} + \Delta m_t = \widetilde{Z}_t,$$

we see that we are in trouble if $\widetilde{Z} = 0$ and $\nu = \Delta m \neq 0$, that is on the set $\{\widetilde{Z} = 0 < Z_-\}$.

Note that

$$d\ell_t = \ell_{t-}\kappa_t dm_t^{\mathbb{G}} = -\ell_{t-} \frac{1}{Z_{t-} + \nu_t} \nu_t dM_t^{\mathbb{G}}$$

is indeed a positive $\mathbb{G}$-local martingale, since $\frac{1}{Z_{t-}+\nu_t}\nu_t < 1$.

**For any (bounded) $X$ satisfying NUPBR$(\mathbb{F})$, $X^\tau$ satisfies NUPBR$(\mathbb{G})$ if and only if the thin set $\Lambda := \left\{\widetilde{Z} = 0 \ \& \ Z_- > 0\right\}$ is evanescent (equivalently, $\eta = \infty$).**

Here $\eta = \zeta \mathbb{1}_{\{\widetilde{Z}_\zeta = 0 < Z_{\zeta-}\}} + \infty \mathbb{1}_{\Lambda^c}$ where $\zeta := \inf\{t : Z_t = 0\}$

The proof in Aksamit et al. is based on the following (new) decomposition: if $X$ is an $\mathbb{F}$-local martingale, the process

$$X_t^{\mathbb{G}} := X_t^\tau - \int_0^{t\wedge\tau} \frac{1}{\widetilde{Z}_s} d[m, X]_s + (\Delta X_\eta \mathbb{1}_{[\![\eta,\infty[\![})^{p,\mathbb{F}}_{t\wedge\tau}, \quad t \geq 0$$

is a $\mathbb{G}$-local martingale.

Then, one defines

$$\ell = \mathcal{E}(-\frac{1}{Z_-} \mathbb{1}_{[0,\tau]} \centerdot m^{\mathbb{G}}).$$

Under the evanescent condition, $\ell$ is a local martingale deflator.

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is a $\mathbb{G}$-local martingale, where $Y^{p,\mathbb{F}}$ is the dual predictable projection.

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**Example where $\left\{\widetilde{Z} = 0 \,\&\, Z_- > 0\right\}$ is not evanescent.**

Let $N$ be a Poisson process (with jump times $T_n$), and $M$ the compensated martingale

$$dS_t = S_{t-}\psi dM_t, \quad S_0 = 1, \qquad M_t := N_t - \lambda t,$$

and

$$\tau = \frac{1}{2}(T_1 + T_2)$$

$$Z_t := \mathbb{P}(\tau > t | \mathcal{F}_t) \quad = \quad 1\!\!1_{\{T_1 > t\}} + 1\!\!1_{\{T_1 \leq t\}} 1\!\!1_{\{T_2 > t\}} e^{-\lambda(t - T_1)}.$$

and $Z = \widetilde{Z}$ (since $\tau$ avoids $\mathbb{F}$ stopping times).

Then $\{\widetilde{Z} = 0 < Z_-\} = [\![T_2]\!]$.

As Johannes (and Bruno) claim

**We have no idea to solve this in general**

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## Case of honest times

A random time $\tau$ is **honest** if $\widetilde{Z}_\tau = 1$. In that case, for any $\mathbb{F}$ martingale $X$ (in particular for $m$ and $S$)

$$X_t^{\mathbb{G}} := X_t^\tau - \int_0^{t\wedge\tau} \frac{d\langle X, m\rangle_s^{\mathbb{F}}}{Z_s} + \int_{t\wedge\tau}^t \frac{d\langle X, m\rangle_s^{\mathbb{F}}}{1 - Z_s}$$

is a $\mathbb{G}$ local martingale.

**Let $\tau$ be a finite honest time and assume that the market $(S^0, S)$ is complete. Then, if $\tau$ is not an $\mathbb{F}$-stopping time, there are classical arbitrages before and after $\tau$.**

If $\tau$ is an honest time avoiding $\mathbb{F}$ stopping times in a continuous filtration in a Brownian filtration (so that the market is complete). Then, $Z_\tau = 1$ and the random variable $\xi := m_\tau - 1 \geq 0$ yields an arbitrage of the first kind .

Let $dm_t = \varphi_t dS_t$ and $\hat{\varphi}_t := -1\!\!1_{\tau<t}\varphi_t$. For all $x > 0$, we have

$$V(x, \hat{\varphi})_\infty = x + m_\tau - m_\infty = x + 1 + \xi > \xi$$

thus showing that $\xi$ is an arbitrage of the first kind.

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**Assume that $\tau$ is a honest time, which satisfies $Z_\tau < 1$ and that all $\mathbb{F}$ martingales are continuous (hence $\tau$ does not avoid $\mathbb{F}$-stopping times). Then, NA1 holds after $\tau$.**

A deflator is given by $d\ell_t = -\frac{\ell_t}{1-Z_t} dm_t^{\mathbb{G}}$.

The proof is based on Itô's calculus and the fact that, for any $\mathbb{F}$ martingale $X$ (in particular for $m$ and $S$)

$$X_t^{\mathbb{G}} := X_t^\tau - \int_0^{t\wedge\tau} \frac{d\langle X, m\rangle_s^{\mathbb{F}}}{Z_s} + \int_{t\wedge\tau}^t \frac{d\langle X, m\rangle_s^{\mathbb{F}}}{1 - Z_s}$$

is a $\mathbb{G}$ local martingale. Looking for a deflator of the form $dL_t = L_t \kappa_t d\widehat{m}_t$, and using integration by parts formula, we obtain that, for $\kappa = -(1 - Z)^{-1}$, the process $L(S - S^\tau)$ is a local martingale.

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# General Results

Suppose that $\tau$ is an honest time such that $Z_\tau < 1$. Then, the following assertions are equivalent.
(a) For any $S$ satisfying NUPBR$(\mathbb{F})$, the process $S - S^\tau$ satisfies NUPBR$(\mathbb{G})$.
(b) The thin set $\{\widetilde{Z} = 1 \ \& \ Z_- < 1\}$ is evanescent.

Same problems can be solved for an initial enlargement, under absolute continuity Jacod's hypothesis (see Acciaio et al. and Aksamit et al.) and for a progressive enlargement, when the absolute continuity Jacod's hypothesis holds.

# REFERENCES

Acciaio B., Fontana C., Kardaras C.(2014) Arbitrage of the First Kind and Filtration Enlargements In Semimartingale Financial Model

Aksamit A., Choulli T., Deng J. and JM. (2013) Non-Arbitrage up to Random Horizon for Semimartingale Models (arkiv)

Aksamit A., Choulli T., Deng J. and JM. (2013) Non-Arbitrage under a Class of Honest Times (arxiv)

Aksamit A., Choulli T., and JM. (2013) Thin random times (working paper)

Aksamit, A., Choulli, T., Deng, J., and JM. (2014) : Arbitrages in a progressive enlargement setting , Arbitrage, Credit and Informational Risks, Peking University Series in Mathematics. Vol. 6, 55-88, World Scientific

Dellacherie, M., Maisonneuve, B. and Meyer, P.A. (1992), Probabilités et Potentiel, chapitres XVII-XXIV: Processus de Markov, Compléments de calcul stochastique, Hermann, Paris.

Fontana, C., JM, Song, S. (2013) On arbitrages arising with honest times, to appear, Finance and Stochastics

Imkeller, P. (2002), Random times at which insiders can have free lunches, Stochastics and Stochastic Reports, 74(1-2): 465–487.

Jeulin, T. (1980), Semi-martingales et Grossissement d'une Filtration, Lecture Notes in Mathematics, vol. 833, Springer, Berlin - Heidelberg - New York.

**Thank you for your attention**

**Let $\tau$ be a finite honest time and assume that the market $(S^0, S)$ is complete. Then, if $\tau$ is not an $\mathbb{F}$-stopping time, there are classical arbitrages before and after $\tau$.**

**Before $\tau$**

From $m = Z + A^o$ and $Z_\tau = 1$, we deduce that $m_\tau \geq 1$.

Since $\tau$ is not a stopping time, $\mathbb{P}(A^o_\tau > 0) > 0$.

The market being complete, the martingale $m_t = 1 + \int_0^t \varphi_s dS_s$ for a predictable $\varphi$. Hence, $V_t = \int_0^t \varphi_s dS_s$ is the value of a self financing portfolio, with initial value 0, and $V_\tau = \int_0^\tau \varphi_s dS_s \geq 0$. One has also $\mathbb{P}(V_\tau > 0) > 0$.

Furthermore, since $m_t \geq 0$, one has $V_t \geq -1$, and the strategy $\varphi$ is admissible.

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Furthermore, since $m_t \geq 0$, one has $V_t \geq -1$, and the strategy $\varphi$ is admissible.

## Classical arbitrages after $\tau$:

Using $m = Z + A^p$, one obtains that, for $t > \tau$, $m_t - m_\tau = Z_t - 1$.

Consider the (finite) $\mathbb{G}$-stopping time

$$\nu := \inf\{s > \tau : Z_s \leq \frac{1}{2}\}.$$

Then,

$$m_\nu - m_\tau = Z_\nu - 1 \leq \frac{-1}{2} \leq 0,$$

and, as $\tau$ is not an $\mathbb{F}$-stopping time,

$$\mathbb{P}(m_\nu - m_\tau < 0) > 0.$$

Hence $-\int_\tau^{t\wedge\nu} \varphi_s dS_s = m_{\tau\wedge t} - m_{t\wedge\nu}$ is the value of a self-financing strategy with initial value 0 and terminal value $m_\tau - m_\nu \geq 0$ satisfying $\mathbb{P}(m_\tau - m_\nu > 0) > 0$.

From $m = Z + A$ and the fact that $A_t = A_{t\wedge\tau}$, one obtains that $m_t - m_\tau = Z_t - Z_\tau \geq -1$, hence the strategy is admissible.

If $\tau$ is an honest time **avoiding $\mathbb{F}$ stopping times in a continuous filtration** in a Brownian filtration (so that the market is complete). Then, $Z_\tau = 1$ and the random variable $\xi := m_\tau - 1 \geq 0$ yields **an arbitrage of the first kind**.

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