Singular Fibrations on 4–Manifolds

Stefan Behrens

Rényi Institute, Budapest

March 24, 2014

Goals for today

1. Motivation and recent history
   - 4-dimensional exoticity
   - What are the Seiberg-Witten invariants?
   - Symplectic and Lefschetz, near and broken
2. The structure of maps from 4–manifolds to surfaces
   - Semi-local aspects
   - Global aspects
   - 1-parameter aspects
3. Where to go from here?
   - Pictorial descriptions of 4-manifolds
   - Some directions and open problems

Without further notice:

- **All manifolds and maps are smooth.**
- **All manifolds are closed, connected, and oriented.**

Motivation and recent history

Dimension 4 is different: the facts

Dimension 4 plays a special roll in manifold topology. It is the border between low and high dimensions. Here is one of the most interesting and still poorly understood phenomena:

Theorem

There are infinite sequences $X_1, X_2, \dots$ of closed, oriented 4–manifolds that are **pairwise homeomorphic***, but* **not diffeomorphic***.*

This cannot happen in any other dimension!

Dimension 4 is different: the methods

- **How to show that two 4-manifolds are homeomorphic?**
  - Try to use the s-cobordism theorem.
  - *But:* The homotopy type has to be nice. ("good groups", . . .)
- **How to tell homeomorphic smooth 4-manifolds apart?**
  - Stare at PDEs, for example the Seiberg-Witten equations:

$$D_A\phi = 0, \quad F_A^+ = \sigma(\phi) + i\eta$$

    (or be very good with Heegaard-Floer theory.)
- **How to show that two 4-manifolds are diffeomorphic?**
  - Main tools: handlebody theory, patience, luck

Dimension 4 is different: the methods

- **How to show that two 4-manifolds are homeomorphic?**
  - Try to use the s-cobordism theorem.
  - *But:* The homotopy type has to be nice. ("good groups",...)
- **How to tell homeomorphic smooth 4-manifolds apart?**
  - Stare at PDEs, for example the Seiberg-Witten equations:

$$D_A\phi = 0, \quad F_A^+ = \sigma(\phi) + i\eta$$

  (or be very good with Heegaard-Floer theory.)
- **How to show that two 4-manifolds are diffeomorphic?**
  - Main tools: handlebody theory, patience, luck

Dimension 4 is different: the methods

- **How to show that two 4-manifolds are homeomorphic?**
  - Try to use the s-cobordism theorem.
  - *But:* The homotopy type has to be nice. ("good groups",...)
- **How to tell homeomorphic smooth 4-manifolds apart?**
  - Stare at PDEs, for example the Seiberg-Witten equations:

$$D_A\phi = 0, \quad F_A^+ = \sigma(\phi) + i\eta$$

    (or be very good with Heegaard-Floer theory.)
- **How to show that two 4-manifolds are diffeomorphic?**
  - Main tools: handlebody theory, patience, luck

Seiberg-Witten invariants and sympletic 4-manifolds

- The Seiberg-Witten equations give rise to a function

$$SW: Spin^c(X) \to \mathbb{Z},$$

the so-called **Seiberg-Witten invariant**.
- A priori, the geometric meaning of SW is mysterious.
- A geometric interpretation is available for *symplectic** 4-manifolds:

Theorem (Taubes, "$SW = Gr$")

For symplectic (X, ω) the value $SW(\mathfrak{s})$ can be expressed as a count of (closed) pseudo-holomorphic curves.

(*symplectic structure: 2-form ω with $d\omega = 0$ and $\omega^2 > 0$)

Seiberg-Witten invariants and sympletic 4-manifolds

- The Seiberg-Witten equations give rise to a function

$$SW: Spin^c(X) \to \mathbb{Z},$$

the so-called **Seiberg-Witten invariant**.

- A priori, the geometric meaning of SW is mysterious.
- A geometric interpretation is available for *symplectic** 4-manifolds:

Theorem (Taubes, "$SW = Gr$")

For symplectic (X, ω) the value $SW(\mathfrak{s})$ can be expressed as a count of (closed) pseudo-holomorphic curves.

(*symplectic structure: 2-form ω with $d\omega = 0$ and $\omega^2 > 0$)

Not all 4–manifolds are symplectic, but. . .

In order to extend his results to general 4-manifolds, **Taubes** suggested the following approach:

- SW is only available for manifolds with $b_2^+ > 0$.
- All these manifolds admit *near-symplectic structures* (i.e. $\omega^2 \geq 0$ and $\omega^2 > 0$ outside a 1-dimensional submanifold)

$\rightsquigarrow$ **Idea:** Count pseudo-holomorphic curves with boundary conditions on the degeneracy loci of near-symplectic forms!

Unfortunately, this appears to be problematic.

Not all 4–manifolds are symplectic, but. . .

In order to extend his results to general 4-manifolds, **Taubes** suggested the following approach:

- SW is only available for manifolds with $b_2^+ > 0$.
- All these manifolds admit *near-symplectic structures* (i.e. $\omega^2 \geq 0$ and $\omega^2 > 0$ outside a 1-dimensional submanifold)

$\rightsquigarrow$ **Idea:** Count pseudo-holomorphic curves with boundary conditions on the degeneracy loci of near-symplectic forms!

Unfortunately, this appears to be problematic.

Not all 4–manifolds are symplectic, but. . .

In order to extend his results to general 4-manifolds, **Taubes** suggested the following approach:

- SW is only available for manifolds with $b_2^+ > 0$.
- All these manifolds admit *near-symplectic structures* (i.e. $\omega^2 \geq 0$ and $\omega^2 > 0$ outside a 1-dimensional submanifold)

$\rightsquigarrow$ **Idea:** Count pseudo-holomorphic curves with boundary conditions on the degeneracy loci of near-symplectic forms!

Unfortunately, this appears to be problematic.

Not all 4–manifolds are symplectic, but. . .

In order to extend his results to general 4-manifolds, **Taubes** suggested the following approach:

- SW is only available for manifolds with $b_2^+ > 0$.
- All these manifolds admit *near-symplectic structures* (i.e. $\omega^2 \geq 0$ and $\omega^2 > 0$ outside a 1-dimensional submanifold)

$\rightsquigarrow$ **Idea:** Count pseudo-holomorphic curves with boundary conditions on the degeneracy loci of near-symplectic forms!

Unfortunately, this appears to be problematic.

Symplectic 4-manifolds and Lefschetz fibrations

- For a topologist, symplectic structures are not the most natural objects to study.
- But their existence can be rephrased in terms of maps to the 2–sphere.

Theorem (Donaldson, Gompf)

A closed 4-manifold X admits a symplectic structure if and only if $X\#n\overline{\mathbb{C}\mathrm{P}^2}$ admits a Lefschetz fibration over S^2 with essential fibers.*

*(*this excludes some torus bundles and blow-ups thereof)*

What is a Lefschetz fibration?

From now on: Let X be a 4-manifold and B a surface.

Definition (Lefschetz fibrations)

A surjective map $f\colon X \to B$ is called a **Lefschetz fibration** if all critical points are of the form

$$(\mathbb{C}^2, 0) \ni \ (z, w) \mapsto z^2 + w^2 \ \in (\mathbb{C}, 0)$$

in local complex charts which respect the orientations.

- In other words, Lefschetz fibrations are locally holomorphic maps with only (complex) non-degenerate critical points.
- Orientation reversing charts $\rightsquigarrow$ **achiral** Lefschetz singularities

What is a Lefschetz fibration?

From now on: Let X be a 4-manifold and B a surface.

Definition (Lefschetz fibrations)

A surjective map $f\colon X \to B$ is called a **Lefschetz fibration** if all critical points are of the form

$$(\mathbb{C}^2, 0) \ni \; (z, w) \mapsto z^2 + w^2 \; \in (\mathbb{C}, 0)$$

in local complex charts which respect the orientations.

- In other words, Lefschetz fibrations are locally holomorphic maps with only (complex) non-degenerate critical points.
- Orientation reversing charts $\rightsquigarrow$ **achiral** Lefschetz singularities

The topology of Lefschetz fibrations

Let $f\colon X \to B$ be a LF and $C_f \subset X$ its critical points.

- The **regular fibers** of f are closed, orientable surfaces of some genus g.
- When approaching a critical value, a s.c.c. in a nearby regular fiber collapses to a point $\rightsquigarrow$ **vanishing cycles**
- This data determines the topology over a neighborhood of a critical value.
- The counterclockwise **monodromy** around a critical value is the Dehn twist about the vanishing cycle

$\rightsquigarrow$ f is determined by $\pi_1(B \setminus f(C_f)) \to \mathrm{Mod}(\Sigma_g)$.

The topology of Lefschetz fibrations

Let $f\colon X \to B$ be a LF and $C_f \subset X$ its critical points.

- The **regular fibers** of f are closed, orientable surfaces of some genus g.
- When approaching a critical value, a s.c.c. in a nearby regular fiber collapses to a point $\rightsquigarrow$ **vanishing cycles**
- This data determines the topology over a neighborhood of a critical value.
- The counterclockwise **monodromy** around a critical value is the Dehn twist about the vanishing cycle

$\rightsquigarrow$ f is determined by $\pi_1(B \setminus f(C_f)) \to \mathrm{Mod}(\Sigma_g)$.

Another geometric interpretation for SW

- Donaldson-Smith and Usher gave another description of SW in terms of language of Lefschetz fibrations.

Theorem (Donaldson-Smith, Usher, "standard surface count")

Let $f : X \to S^2$ be a Lefschetz fibration. Then SW is a count of pseudo-holomorphic multi-sections of f.

- Multi-sections of Lefschetz fibrations are accessible by mapping class group methods.

($\rightsquigarrow$ recent preprint of Baykur-Hayano)

How Lefschetz fibrations were broken

Auroux, Donaldson, and Katzarkov extended Donaldson's existence proof of LFs to the near-symplectic setting.

- **Observation:** degeneracy loci $\rightsquigarrow$ indefinite fold singularities

Definition (Broken Lefschetz fibrations)

A **broken Lefschetz fibration** is a surjective smooth map $X \to B$ whose critical points are Lefschetz singularities or indefinite folds.

Theorem (Auroux-Donaldson-Katzarkov)

A 4-manifolds admits a near-symplectic structure if and only if some blow-up admits a broken Lefschetz fibration with essential fibers.

How Lefschetz fibrations were broken

Auroux, Donaldson, and Katzarkov extended Donaldson's existence proof of LFs to the near-symplectic setting.

- **Observation:** degeneracy loci $\rightsquigarrow$ indefinite fold singularities

Definition (Broken Lefschetz fibrations)

A **broken Lefschetz fibration** is a surjective smooth map $X \to B$ whose critical points are Lefschetz singularities or indefinite folds.

Theorem (Auroux-Donaldson-Katzarkov)

A 4-manifolds admits a near-symplectic structure if and only if some blow-up admits a broken Lefschetz fibration with essential fibers.

How Lefschetz fibrations were broken

Auroux, Donaldson, and Katzarkov extended Donaldson's existence proof of LFs to the near-symplectic setting.

- **Observation:** degeneracy loci $\rightsquigarrow$ indefinite fold singularities

Definition (Broken Lefschetz fibrations)

A **broken Lefschetz fibration** is a surjective smooth map $X \to B$ whose critical points are Lefschetz singularities or indefinite folds.

Theorem (Auroux-Donaldson-Katzarkov)

A 4-manifolds admits a near-symplectic structure if and only if some blow-up admits a broken Lefschetz fibration with essential fibers.

And what about SW?

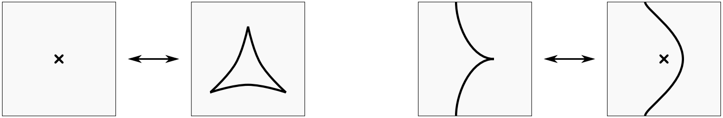
Based on the Donaldson-Smith approach, **Perutz** constructed his "Lagrangian Matching Invariants".

- The input is a pair (X, f), where $f\colon X \to S^2$ is a BLF.
- The 'invariants' behave much like SW.
- Unfortunately, the proof of invariance got stuck.

$\rightsquigarrow$ There is still no fully established geometric interpretation of SW for all 4-manifolds with $b_2^+ > 0$.

Perutz's work led **Lekili** to study homotopies between BLFs.

- Lefschetz singularities are not stable under perturbations, they dissolve into indefinite folds and cusps.
- Lekili: Lefschetz + ind. folds $\rightsquigarrow$ ind. folds + cusps



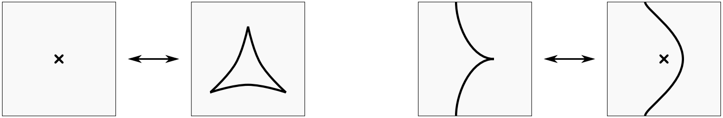
Definition (Wrinkled fibrations, preliminary version)

A surjective smooth map $f : X \to B$ is called a **wrinkled fibration** if all of its critical points are indefinite folds and cusps.

- X admits a BLF over $B \iff X$ admits a WF over B

Perutz's work led **Lekili** to study homotopies between BLFs.

- Lefschetz singularities are not stable under perturbations, they dissolve into indefinite folds and cusps.
- Lekili: Lefschetz $+$ ind. folds $\leftrightsquigarrow$ ind. folds $+$ cusps



Definition (Wrinkled fibrations, preliminary version)

A surjective smooth map $f\colon X \to B$ is called a **wrinkled fibration** if all of its critical points are indefinite folds and cusps.

- X admits a BLF over $B \iff X$ admits a WF over B

...and then there were all 4-manifolds...

The Lefschetz-cusp exchange had a side effect.

- It was known that S^4 and $S^1 \times S^3$ admit BLFs over S^2.
- These manifolds are not near-symplectic

$\rightsquigarrow$ **Could it be that all 4-manifolds admit BLFs over S^2?**

Theorem (Akbulut-Karakurt, Baykur, Lekili, Gay-Kirby, (Saeki))

Yes!

. . . and then there were all 4-manifolds. . .

The Lefschetz-cusp exchange had a side effect.

- It was known that S^4 and $S^1 \times S^3$ admit BLFs over S^2.
- These manifolds are not near-symplectic

$\rightsquigarrow$ **Could it be that all 4-manifolds admit BLFs over S^2?**

Theorem (Akbulut-Karakurt, Baykur, Lekili, Gay-Kirby, (Saeki))

Yes!

. . . and then there were all 4-manifolds. . .

The Lefschetz-cusp exchange had a side effect.

- It was known that S^4 and $S^1 \times S^3$ admit BLFs over S^2.
- These manifolds are not near-symplectic

$\rightsquigarrow$ **Could it be that all 4-manifolds admit BLFs over S^2?**

Theorem (Akbulut-Karakurt, Baykur, Lekili, Gay-Kirby, (Saeki))

Yes!

So maps with folds and cusps, you say?

Well, those were already studied 50 years ago!

The structure of maps from 4–manifolds to surfaces

The outside-in approach

Instead of starting with a special situation and trying to generalize, let's look at everything (all 4-manifolds and all maps to surfaces) and try to specialize.

- General smooth maps $X \to B$ are complicated.

$\rightsquigarrow$ Find properties which are **dense**, ideally **open** and **stable under perturbations**.

- Focus on what's left and discard the rest.

- **Note:** Morse functions that are injective on their critical points satisfy these properties. (Cerf: "excellent functions")

The outside-in approach

Instead of starting with a special situation and trying to generalize, let's look at everything (all 4-manifolds and all maps to surfaces) and try to specialize.

- General smooth maps $X \to B$ are complicated.

$\rightsquigarrow$ Find properties which are **dense**, ideally **open** and **stable under perturbations**.

- Focus on what's left and discard the rest.

- **Note:** Morse functions that are injective on their critical points satisfy these properties. (Cerf: "excellent functions")

Definition (Excellent maps, wrinkled fibrations)

A map $f\colon X \to B$ is called **excellent** if all of its critical values are of the following form:

- Folds: $$(t,x,y,z) \mapsto (t, x^2 + y^2 \pm z^2)$$
- Cusps: $$(t,x,y,z) \mapsto (t, x^3 + 3tx + y^2 \pm z^2)$$
- Double folds: $$\begin{array}{rcl}(t,x,y,z) & \mapsto & (t,\ x^2+y^2 \pm z^2)\\ (t',x',y',z') & \mapsto & (x'^2+y'^2 \pm z'^2,\ t')\end{array}$$

Fold and cusps can be **definite** $(+)$ and **indefinite** $(-)$.
A **wrinkled fibration** is an excellent map without definite singularities.

- The terminology/author ratio in this context is rather high. Other names: generic maps, stable maps, Thom-Boardman maps with normal crossings, Morse 2-functions, . . .

Theorem

The set of excellent maps $X \to B$ is dense, open, and stable under small perturbations in $C^\infty(X, B)$.

Some keywords for the "canonical" proof of this:

- Multi-germ codimension, stability
- Mather's "nice dimensions"
- Multi-jet transversality theorem

More about folds

$$(t, x, y, z) \mapsto (t, x^2 + y^2 \pm z^2)$$

- trivial family of 3-dimensional Morse singularities (fancy: constant 1-dimensional unfolding of 3d Morse)

$\rightsquigarrow$ folds appear in 1-parameter families.

- the horizontal projection of vertical (t-)slices is Morse

- Note: index k and index $n - k$ are interchanged by the diffeos $(t, x, y, z) \mapsto (-t, x, y, -z)$ and $(u, v) \mapsto (-u, -v)$

More about cusps

$$(t, x, y, z) \mapsto (t, x^3 + 3tx + y^2 \pm z^2)$$

- standard model for cancellation of critical points in 3d Morse functions (fancy: miniversal unfolding of 3d birth-death)
- the horizontal projection of all vertical slices but one is Morse
- Cusps are isolated in the critical locus and are always surrounded by folds.

What you should keep in mind

Excellent maps

from 4-manifolds to surfaces

locally look like

excellent 1–parameter families of functions on 3-manifolds!

The topology of excellent maps

Let's take a closer look at the global structure of an excellent map $f\colon X \to B$.

- $C_f \subset X$ is a 1–dimensional submanifold
  $\rightsquigarrow$ finite collection of embedded circles
- $f|_{C_f}$ is an immersion with cusps and normal crossings
  $\rightsquigarrow$ $f(C_f) \subset B$ is a topologically embedded graph with vertices of valence 2 (cusps) and 4 (double folds)
- The regular fibers of f are closed surfaces, possibly disconnected.

The topology of excellent maps

Let's take a closer look at the global structure of an excellent map $f\colon X \to B$.

- $C_f \subset X$ is a 1–dimensional submanifold
  $\rightsquigarrow$ finite collection of embedded circles
- $f|_{C_f}$ is an immersion with cusps and normal crossings
  $\rightsquigarrow$ $f(C_f) \subset B$ is a topologically embedded graph with vertices of valence 2 (cusps) and 4 (double folds)
- The regular fibers of f are closed surfaces, possibly disconnected.

Reference arcs

- A **reference arc** for f is an embedded arc $\gamma \subset B$ such that:
  - The endpoints are regular values.
  - γ does not meet the cusps and double points
  - γ is transverse to the fold arcs

Theorem

Let $\gamma\colon [0,1] \to B$ be an embedding of a reference arc. Then $f^{-1}(\operatorname{im}\gamma)$ is a smooth 3-manifold and the composition

$$f_\gamma\colon f^{-1}(\operatorname{im}\gamma) \xrightarrow{f} \operatorname{im}\gamma \xrightarrow{\gamma^{-1}} [0,1]$$

is a Morse function.

- The ascending and descending spheres of f_γ are called the **vanishing sets** of γ.

"Surface valued Morse theory"

Idea:

- Fix a reference fiber over each region.
- For each fold arc connect the two adjacent regions with a reference arc.
- Record the vanishing sets and some additional gluing data.

($\rightsquigarrow$ Gay-Kirby: Reconstructing 4-manifolds. . .)

Problems:

- regions that are not disks $\rightsquigarrow$ monodromy
- tori and spheres in fibers $\rightsquigarrow$ gluing ambiguities

Simplifying excellent maps

Now that we have an abundance of maps with good local behavior, let's try to simplify them as much as possible.

Theorem (Gay-Kirby, Saeki)

Let $f\colon X \to B$ be continuous and let $I_f \in \mathbb{N} \cup \{\infty\}$ be the index of $f_*(\pi_1(X))$ in $\pi_1(B)$.

1. *f is homotopic to a wrinkled fibration if and only I_f is finite.*
2. *If $I_f = 1$, then f can be further homotoped to a map with connected fibers.*

- In particular, this always holds for $B = S^2$ ($\rightsquigarrow$ Saeki)

Simplifying excellent maps

Now that we have an abundance of maps with good local behavior, let's try to simplify them as much as possible.

Theorem (Gay-Kirby, Saeki)

Let $f \colon X \to B$ be continuous and let $I_f \in \mathbb{N} \cup \{\infty\}$ be the index of $f_*(\pi_1(X))$ in $\pi_1(B)$.

1. *f is homotopic to a wrinkled fibration if and only if I_f is finite.*
2. *If $I_f = 1$, then f can be further homotoped to a map with connected fibers.*

- In particular, this always holds for $B = S^2$ ($\rightsquigarrow$ Saeki)

Simple Wrinkled Fibrations

Definition (Simple wrinkled fibrations)

A wrinkled fibration $X \to S^2$ is called **simple** if it is injective on its critical locus (and has at least one cusp).

Simple wrinkle fibrations are indeed very simple:

- There are only two regions of regular values.
- Both regions are disks.

Theorem (Williams)

Every map $X \to S^2$ is homotopic to a simple wrinkled fibration of arbitrarily high fiber genus.

Simple Wrinkled Fibrations

Definition (Simple wrinkled fibrations)

A wrinkled fibration $X \to S^2$ is called **simple** if it is injective on its critical locus (and has at least one cusp).

Simple wrinkle fibrations are indeed very simple:

- There are only two regions of regular values.
- Both regions are disks.

Theorem (Williams)

Every map $X \to S^2$ is homotopic to a simple wrinkled fibration of arbitrarily high fiber genus.

The 1-parametric theory

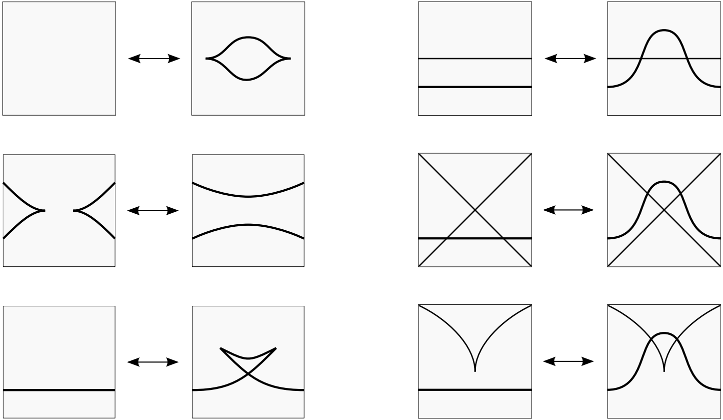
- The set of excellent maps is not connected. It is the top stratum of a Cerf-style stratification of $C^\infty(X, B)$.
- Families of excellent maps will generically meet higher codimension strata.

$\rightsquigarrow$ "catastrophes"

- **Note:** $C^\infty(X, B)$ itself is not connected for closed B. (for $B = S^2$ see Kirby-Melvin-Teichner)

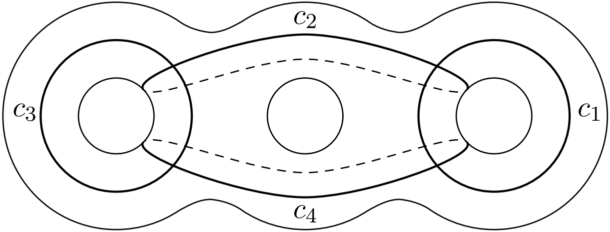
The 1-parameter "moves" for excellent maps

1-parameter catastrophes as seen in the target:



Where to go from here?

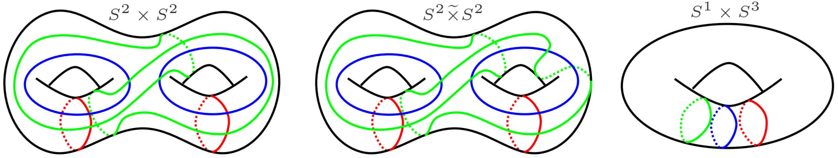
Surface diagrams (Williams)



$$\mathfrak{S} = (\Sigma_g; c_1, \ldots, c_l), \quad i(c_i, c_{i+1}) = 1 \text{ where } l+1=1$$

- Derived from simple wrinkled fibrations over S^2.

Trisection diagrams (Gay-Kirby)



$$(\Sigma_g; \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$$

- $\boldsymbol{\alpha} = \{\alpha_1, \ldots, \alpha_g\}$, $\boldsymbol{\beta} = \ldots$, $\boldsymbol{\gamma} = \ldots$ as in Heegaard diagrams
- $(\Sigma_g; \boldsymbol{\alpha}, \boldsymbol{\beta})$, $(\Sigma_g; \boldsymbol{\beta}, \boldsymbol{\gamma})$, $(\Sigma_g; \boldsymbol{\gamma}, \boldsymbol{\alpha})$ Heegaard diagrams of $\natural^k(S^1 \times S^2)$ for some k
- Derived from certain Morse 2-functions to $\mathbb{R}^2$.

Uniqueness?

Trisection diagrams:

- There is a uniqueness statement similar to that for Heegaard diagrams.
- This is possible since $C^\infty(X, \mathbb{R}^2)$ is contractible.

Surface diagrams: (more complicated)

- First obstacle: $C^\infty(X, S^2)$ is usually not connected.
- There are four moves for homotopic diagrams. ($\rightsquigarrow$ Williams, Hayano, B.-Hayano)
- So far, there are no moves to jump between different homotopy classes.

Smooth 4-manifold invariants?

- Relations to known invariants?
- Definition of new invariants?

Cerf theory?

- How uniquely are the 1-parameter catastrophes realized?

Thank you!

. . . and sorry for going over time