An Introduction to Symplectic and Contact Topology and the Technique of Generating Families

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March 2014
1 Introduction to Symplectic and Contact Topology
Outline

1. Introduction to Symplectic and Contact Topology
2. Introduction to Generating Families
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1. Introduction to Symplectic and Contact Topology

2. Introduction to Generating Families

3. Invariants from Generating Families
   - Generating Family Homology for a Legendrian Submanifold
   - Wrapped Generating Family Cohomology for a Lagrangian Cobordism
Where Are We?

1. Introduction to Symplectic and Contact Topology

2. Introduction to Generating Families

3. Invariants from Generating Families
   - Generating Family Homology for a Legendrian Submanifold
   - Wrapped Generating Family Cohomology for a Lagrangian Cobordism
Important Symplectic and Contact Objects

Symplectic Manifold \((X^{2n}, \omega)\)

**Symplectomorphism**
\[\psi^* \omega = \omega\]

Lagrangian Submanifold
\[L^n : \omega|_{TL} \equiv 0\]

Contact Manifold \((Y^{2n+1}, \xi)\)

**Contactomorphism**
\[\kappa^* \xi = \xi\]

Legendrian Submanifold
\[\Lambda^n : T\Lambda \subset \xi\]
Symplectic Manifolds: \((X^{2n}, \omega)\)

\(\omega\) is a closed, non-degenerate 2-form:

\[d \omega = 0, \quad \omega^n \text{ is a volume form.}\]
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Example

- Any orientable surface \(\Sigma\) with area form \(\omega\);

\[\text{S}^4\text{ is not a symplectic manifold: } H_2(S^4) = 0.\]
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- \((\mathbb{R}^{2n}, \omega_0 = dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n)\);
- Cotangent Bundle, \(T^*M\), has a canonical 1-form \(\lambda_0\), \(\omega_0 = d\lambda_0\) is a symplectic form.
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\(S^4\) is not a symplectic manifold: \(H^2(S^4) = 0\).

Theorem (Darboux’s Theorem)

All symplectic manifolds are locally equivalent to \((\mathbb{R}^{2n}, \omega_0)\).
Symplectic Diffeomorphisms: $\omega(v, w) = \omega(\psi_* v, \psi_* w)$.

Example ($\mathcal{A}_2$, $\mathcal{A}$): Any area-preserving transformation; ($\mathbb{R}^2$, $\mathcal{A}_0$): Products of area-preserving transformations of the $x_i$-$y_i$-planes; ($\mathbb{R}^2$, $\mathcal{A}$): Hamiltonian Functions and Vector Fields

For any smooth $H_t: \mathbb{R} \to \mathbb{R}$, define $v_t$ by $v_t(v, \cdot) = dH_t$; The flow of $v_t$ defines a symplectic isotopy $t : (X, \mathcal{A}) \to (X, \mathcal{A})$. 

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The flow of $v_t$ defines a symplectic isotopy $\psi_t : (X, \omega) \to (X, \omega)$. 
Lagrangian Submanifolds: \( L^n : \omega|_{TL} \equiv 0 \)

Example \((\mathbb{R}^2, \omega)\): Any embedded curve is Lagrangian;

\((\mathbb{R}^n, \omega)\): The \(x_1 x_2 x_3 \ldots x_n\)-plane is Lagrangian; the \(x_1 y_1 x_3 \ldots x_n\)-plane is not Lagrangian;

\((T^* M, d\omega)\): For any function \(f: M \to \mathbb{R}\), the section \(df \to T^* M\) is a Lagrangian submanifold.

There is no embedded Lagrangian \(S^2\) in \(\mathbb{R}^4\).
Example

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- $(T^*M, d\lambda_0)$: For any function $f : M \to \mathbb{R}$,
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! There is no embedded Lagrangian $S^2$ in $\mathbb{R}^4$. 

Contact Manifolds: \((Y^{2n+1}, \xi)\)

\(\xi\) is a field of maximally non-integrable tangent hyperplanes:

\[\xi = \ker \alpha \implies \alpha \wedge (d\alpha)^{n} \neq 0.\]
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- \((\mathbb{R}^{2n+1}, \xi_0 = \ker (dz - \sum y_i dx_i))\);
- (1-Jet Bundles) \(J^1 M = T^* M \times \mathbb{R}\) has a canonical \(\xi_0\);
Contact Manifolds: $(Y^{2n+1}, \xi)$

$\xi$ is a field of maximally non-integrable tangent hyperplanes:

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Theorem (Darboux’s Theorem)

All contact manifolds are locally equivalent to \((\mathbb{R}^{2n+1}, \xi_0)\).
Local and Global Equivalence of Contact Structures

Theorem (Darboux’s Theorem)

All contact manifolds are locally equivalent to $(\mathbb{R}^{2n+1}, \xi_0)$.

Important Problem: Understand the different contact structures on a fixed smooth manifold.

$\xi_0, \xi_1 : \xi_0$ and $\xi_1$ are homotopic as plane fields,

$\mathcal{A}\varphi : (Y, \xi_0) \to (Y, \xi_1)$. 
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\[ \forall \phi : (Y, \xi_0) \rightarrow (Y, \xi_1). \]

Theorem (Gray Stability)

If \(\xi_t\) is a smooth family of contact structures on a closed manifold \(Y\), then there is an isotopy \(\kappa_t : (Y, \xi_0) \rightarrow (Y, \xi_t)\) with \((\kappa_t)_*\xi_0 = \xi_t\).
Tight vs. Overtwisted \((Y^3, \xi)\)

A contact 3-manifold is **overtwisted** if it contains an **overtwisted disk**, \(D\): interior of \(D\) is transversal to \(\xi\) except at 1 point; \(\partial D\) is tangent to \(\xi\).
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**Theorem (Eliashberg)**

*Two overtwisted structures on $Y$ are isotopic if they are homotopic as plane fields.*
Contact Diffeomorphisms: $\kappa_* \xi = \xi$

**Contact Vector Field**: flow preserves the contact structure

contact vector field $\mapsto$ contact isotopy
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**Contact Vector Field:** flow preserves the contact structure

contact vector field $\longrightarrow$ contact isotopy

**Example**

- **Reeb Vector Field:** If $\xi = \ker \alpha$, the Reeb vector field $R_\alpha$ defined by:

$$R_\alpha \in \ker d\alpha, \quad \alpha(R_\alpha) = 1,$$

is a contact vector field.
Contact Diffeomorphisms: \( \kappa_* \xi = \xi \)

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- **Contact Hamiltonian Vector Field**: For any smooth \( H : Y \to \mathbb{R} \), there is a unique vector field \( K_H \subset \xi = \ker \alpha \) so that

\[
V_H = HR_\alpha + K_H
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**Contact Vector Field**: flow preserves the contact structure

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Also: time-dependent Hamiltonian functions, $H_t$, and vector fields $V_{H_t}$
Legendrians are abundant!
Legendrian Submanifolds: $T^n \subset \xi$

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- In higher dimensions, any closed sphere, torus, or $n$-manifold that satisfies certain homotopy-theoretic conditions can be $C^0$-approximated by a Legendrian.
Legendrian Submanifolds: $\mathcal{T} \Lambda^n \subset \xi$

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Interested in the classification of Legendrians up to Legendrian isotopy (equivalently, up to ambient contact isotopy).
Basic Examples of Legendrian Curves

\((\mathbb{R}^3, \ker(dz - ydx))\), \(\Lambda_f = \text{Legendrian lift of the graph of } f : \mathbb{R} \to \mathbb{R}\).
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\Lambda_f = \left\{ \left( x, \frac{df}{dx}(x), f(x) \right) \right\} \subset \mathbb{R}^3.
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More Complicated Legendrian Shapes

- Lift of the graph of this "multi-valued" function to $\mathbb{R}^3$
- No vertical tangents allowed;
- Lift is smooth (semi-cubical cusps);
- At crossing, branch with lesser slope is on top.

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Legendrian Knots

Every topological knot/link has a Legendrian representative.

+ Trefoil

- Trefoil

Figure 8

5₁

5₂
Legendrian sphere in $\mathbb{R}^5$: 
Legendrian Surfaces

Legendrian sphere in $\mathbb{R}^5$:

More possibilities for singularities:
Important Problem: Understand the equivalence of Legendrian submanifolds
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- ($n \geq 4$ and even) Relative invariant $tb^*(\Lambda_1, \Lambda_2) \in \mathbb{Z}/2\mathbb{Z}$. 
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- ( $n \geq 4$ and even) Relative invariant $tb^*(\Lambda_1, \Lambda_2) \in \mathbb{Z}/2\mathbb{Z}$.

When $\Lambda = S^{2n}$, $r(\Lambda) = 0$, $tb(\Lambda)$ determined from $\chi(\Lambda)$. 
Question: Is it possible to construct non-classical invariants of Legendrian submanifolds from the Reeb chords?
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Techniques to Construct Invariants from Reeb Chords

J-Holomorphic Curves
Initiated by Gromov, ’85

Generating Families of Functions
Classic technique;
Modernized by Sikorav, Laudenbach, Chaperon, Viterbo, 80’s and 90’s
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Arguments apply to all dimensions.
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Combinatorics as a bridge.
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Idea of Generating Families

Manifolds of Focus:

**Contact**

$\mathbb{R}^{2n+1}$

\[ J^1(M) = T^*M \times \mathbb{R} \]

**Symplectic**

$\mathbb{R}^{2n}$

\[ T^*(M) \]
Idea of Generating Families

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- In $J^1(M)$, describe Legendrian submanifolds as the “1-jet” of functions $F : M \times \mathbb{R}^N \to \mathbb{R}$. 

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Idea of Generating Families

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- In $T^*(M)$, describe Lagrangian submanifolds as the "derivatives" of functions $F : M \times \mathbb{R}^N \rightarrow \mathbb{R}$. 

Strategy:

- Apply analysis/Morse theoretic arguments to these functions to obtain invariants of the Lagrangian and Legendrian submanifolds.
Idea of Generating Families

Manifolds of Focus:

<table>
<thead>
<tr>
<th>Contact</th>
<th>Symplectic</th>
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<tbody>
<tr>
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- In $J^1(M)$, describe Legendrian submanifolds as the "1-jet" of functions $F : M \times \mathbb{R}^N \to \mathbb{R}$.

- In $T^*(M)$, describe Lagrangian submanifolds as the "derivatives" of functions $F : M \times \mathbb{R}^N \to \mathbb{R}$.

**Strategy:** Apply analysis/Morse theoretic arguments to these functions to obtain invariants of the Lagrangian and Legendrian submanifolds.
Basic Examples

\((\mathbb{R}^3, \ker(dz - ydx))\), \(\Lambda_f = \text{Legendrian lift of the graph of } f: \mathbb{R} \rightarrow \mathbb{R}.\)

\[\Lambda_f = \left\{ \left( x, \frac{df}{dx}(x), f(x) \right) \right\} \subset \mathbb{R}^3.\]
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\(f : \mathbb{R} \rightarrow \mathbb{R}\) "generates" \(\Lambda_f\).
$\Lambda$ = lift of the graph of this “multi-valued" function to $\mathbb{R}^3 = J^1\mathbb{R}$

$\Lambda$ is not the 1-jet of $f : \mathbb{R} \to \mathbb{R}$. 
Λ = lift of the graph of this “multi-valued” function to $\mathbb{R}^3 = J^1\mathbb{R}$

Λ is not the 1-jet of $f : \mathbb{R} \to \mathbb{R}$.

BUT, Λ can be viewed as the “1-jet” of a 1-parameter family of functions

$$F : \mathbb{R} \times \mathbb{R}^N = \{(x, e)\} \to \mathbb{R}.$$
**Idea:** Construct a family of functions $F_x : \mathbb{R}^N = \{e\} \to \mathbb{R}$, for $x \in \mathbb{R}$. 

![Diagram of functions $F_x$](image-url)
**Idea:** Construct a family of functions $F_x : \mathbb{R}^N = \{e\} \to \mathbb{R}$, for $x \in \mathbb{R}$.

This family of functions "generates" $\Lambda$.

\[ \exists F : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R} \text{ so that } \]
\[ \Lambda = \left\{ \left( x, \frac{\partial F}{\partial x}(x, e), F(x, e) \right) : \frac{\partial F}{\partial e}(x, e) = 0 \right\}. \]
Explicit Construction

**Idea:** Construct a family of functions $F_x : \mathbb{R}^N = \{e\} \to \mathbb{R}$, for $x \in \mathbb{R}$.

$\exists F : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ so that

$$\Lambda = \left\{ \left( x, \frac{\partial F}{\partial x}(x, e), F(x, e) \right) : \frac{\partial F}{\partial e}(x, e) = 0 \right\}.$$

$\exists$ choices: Can

- change $N$ by stabilizing with a quadratic: $\tilde{F}(x, e, \tilde{e}) = F(x, e) + Q(\tilde{e});$
- apply a fiber-preserving diffeomorphism: $\tilde{F}(x, e) = F(x, \phi_x(e)).$
GF Domain has Finite Dimension

\[ \exists F : \mathbb{R} \times \mathbb{R}^1 \rightarrow \mathbb{R} \]

that generates \( \Lambda \subset \mathbb{R}^3 = J^1 \mathbb{R} \).
GF Domain has Finite Dimension

\[ \exists F : \mathbb{R} \times \mathbb{R}^1 \to \mathbb{R} \]

that generates \( \Lambda \subset \mathbb{R}^3 = J^1\mathbb{R} \). But,

\[ \exists F : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R} \).

In general, for \( \Lambda \subset J^1(M) \), consider \( F : M \times \mathbb{R}^N \to \mathbb{R} \) for large \( N \).
GF Domain has Finite Dimension

$\exists F : \mathbb{R} \times \mathbb{R}^1 \to \mathbb{R}$

that generates $\Lambda \subset \mathbb{R}^3 = J^1 \mathbb{R}$. But,

$\exists F : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$.

In general, for $\Lambda \subset J^1 (M)$, consider $F : M \times \mathbb{R}^N \to \mathbb{R}$ for large $N$.

Usually want $F$ “nice” (quadratic or linear) outside a compact set.
Generating Families for Lagrangian Submanifolds

Similar set up for Lagrangians:

\[
\exists F : \mathbb{R}^n \to \mathbb{R} \text{ so that } L = \{(x, \frac{\partial F}{\partial x}(x)) : x \in \mathbb{R}^n\}.
\]
Generating Families for Lagrangian Submanifolds

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\[ \exists F : \mathbb{R}^n \to \mathbb{R} \text{ so that } L = \{ (x, \frac{\partial F}{\partial x}(x)) : x \in \mathbb{R}^n \}. \]

\[ \exists F : \mathbb{R}^n \times \mathbb{R}^N \to \mathbb{R} \text{ so that } L = \{ (x, \frac{\partial F}{\partial x}(x, e)) : \frac{\partial F}{\partial e}(x, e) = 0 \}. \]
Important Points

- Not all Legendrian (Lagrangian) submanifolds have generating families.

Sabloff, Fuchs, Pushkar, and others show that for $\Lambda^1 \subset \mathbb{R}^3$:

$\exists$ generating family $\iff \exists$ augmentation of the holomorphic-$DGA$. 
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Sabloff, Fuchs, Pushkar, and others show that for $\Lambda^1 \subset \mathbb{R}^3$:

$\exists$ generating family $\iff \exists$ augmentation of the holomorphic-$DGA$.

If a Legendrian (Lagrangian) submanifold has a generating family and $\kappa_t$ is a contact (symplectic) isotopy, then $\kappa_1(\Lambda)$ has a generating family. "Persistence / Serre Fibration"
Where Are We?

1. Introduction to Symplectic and Contact Topology

2. Introduction to Generating Families

3. Invariants from Generating Families
   - Generating Family Homology for a Legendrian Submanifold
   - Wrapped Generating Family Cohomology for a Lagrangian Cobordism
Given a generating family $f : M \times \mathbb{R}^N \to \mathbb{R}$ for $\Lambda \subset J^1 M$, define the difference function $\delta : M \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ by:

$$\delta(x, e, \tilde{e}) = f(x, \tilde{e}) - f(x, e).$$
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Critical points of $\delta$ are of two types:
Difference Functions of Generating Families

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Critical points of \( \delta \) are of two types:

- For each Reeb chord of \( \Lambda \), there are two critical points of \( \delta \) with critical values \( \pm \) (Reeb chord length);
- There is a non-degenerate critical submanifold diffeomorphic to \( \Lambda \) with critical value 0.
Given a generating family $f : M \times \mathbb{R}^N \to \mathbb{R}$ for $\Lambda \subset J^1 M$, define the difference function $\delta : M \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ by:

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- For each Reeb chord of $\Lambda$, there are two critical points of $\delta$ with critical values $\pm$ (Reeb chord length);
- There is a non-degenerate critical submanifold diffeomorphic to $\Lambda$ with critical value 0.

As Legendrian $\Lambda$ is isotoped, critical values born/die/change.

Apply Morse-theoretic constructions to get an invariant of $\Lambda$ from $\delta$. 

For $\omega \gg 0$, $H_*(\delta^\omega, \delta^{-\omega}) = 0$, for all $\ast$. 
Generating Family Homology and Cohomology

For $\omega \gg 0$, $H_*(\delta^\omega, \delta^{-\omega}) = 0$, for all $\ast$.

Choose $\epsilon$ sufficiently small:

**Definition (LT, Fuchs-Rutherford, Sabloff-LT)**

\[
\begin{align*}
GH_k(f) & = H_{k+N+1}(\delta^\omega, \delta^{\epsilon}), & \widehat{GH}_k(f) & = H_{k+N+1}(\delta^\omega, \delta^{-\epsilon}), \\
GH^k(f) & = H^{k+N+1}(\delta^\omega, \delta^{\epsilon}), & \widehat{GH}^k(f) & = H^{k+N+1}(\delta^\omega, \delta^{-\epsilon}).
\end{align*}
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**Theorem (LT, Fuchs-Rutherford)**

\{ $GH_*(f) : f$ generates $\wedge$ \} is an invariant of $\wedge$. 

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Symplectic and Contact Topology  
Banff 2013  30 / 40
Generating Family Homology and Cohomology

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**Theorem (LT, Fuchs-Rutherford)**

\[ \{ GH_*(f) : f \text{ generates } \wedge \} \text{ is an invariant of } \wedge. \]
\[ \{ GH^*(f) \}, \{ \widehat{GH}_*(f) \}, \{ \widehat{GH}^*(f) \} \text{ are also invariants of } \wedge. \]
Generating Family Polynomials

Work over a field $\mathbb{F}$ and encode $GH_\ast(f)$ by its Poincaré polynomial:

$$\Gamma_f(t) = \sum_i \dim GH_i(f) t^i$$
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Work over a field $\mathbb{F}$ and encode $GH_*(f)$ by its Poincaré polynomial:

$$\Gamma_f(t) = \sum_i \dim GH_i(f) t^i$$

Example

There are two different Legendrian $m(5_2)$ knots with the same $tb$ and $r$ values:

$$P = \{ t^{-2} + t + t^2 \} \quad P = \{ 2 + t \}$$
Duality for Generating Family Homology

There is a structure to the gf-polynomials:

\[ t + 4t^2, \quad 4t^{-2} + 4t^2 \]

cannot occur as gf-polynomials for \( \Lambda^1 \subset \mathbb{R}^3 \).
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cannot occur as gf-polynomials for \( \Lambda^1 \subset \mathbb{R}^3 \).

**Duality Theorem:**

Theorem (Sabloff-LT)

There is a long exact sequence:

\[ \cdots \rightarrow GH^{k-1}(f) \rightarrow GH_{n-k}(f) \rightarrow H^k(\Lambda) \rightarrow \cdots \]
Open Problems for GF-Homology

Important Directions:

- Combinatorially calculate GF-Homology;
- Construct GF-DGA.
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- Combinatorially calculate GF-Homology;
- Construct GF-DGA.

[Henry and Rutherford] A combinatorial-DGA for Legendrian knots motivated by generating families
\[ \Lambda_- \prec_L \Lambda_+ \]

denotes an exact Lagrangian cobordism that is cylindrical over \( \Lambda_\pm \) at \( \pm \infty \).
Lagrangian Floer Cohomology: For closed Lagrangians, define a cohomology with underlying cochain complex the intersection points of the Lagrangians.
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**Wrapped Floer Cohomology:**
[Abbondodolo-Schwarz, Fukaya-Seidel-Smith, Abouzaid-Smith] For Lagrangian cobordisms, cochain complex generated by intersection between the compact pieces of two Lagrangian cobordisms *and* the Reeb chords at the Legendrian ends.
Lagrangian Floer Cohomology: For closed Lagrangians, define a cohomology with underlying cochain complex the intersection points of the Lagrangians.

Wrapped Floer Cohomology:


How to study with generating families?
Assume Legendrians and Lagrangian can be described by compatible generating families:

\[(\Lambda_-, f_-) \prec_{(L, F)} (\Lambda_+, f_+).\]
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\[\exists \text{ generating families}\]

- \(f_\pm : \mathbb{R}^n \times \mathbb{R}^N \to \mathbb{R}\) for \(\Lambda_\pm\) (linear-at-\(\infty\)); and
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\[(\Lambda_-, f_-) \prec_{(L,F)} (\Lambda_+, f_+).\]

\[
\psi : \mathbb{R} \times \mathbb{R}^{2n+1} \rightarrow T^*(\mathbb{R}^+ \times \mathbb{R}^n)
\]

\[
\bar{L} \mapsto \psi(\bar{L})
\]

\exists \text{ generating families}

- \(f_\pm : \mathbb{R}^n \times \mathbb{R}^N \rightarrow \mathbb{R}\) for \(\Lambda_\pm\) (linear-at-\(\infty\)); and
- \(F : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^N \rightarrow \mathbb{R}\) for \(\psi(\bar{L}) \subset T^*(\mathbb{R}^+ \times \mathbb{R}^n)\) that correlates to \(f_\pm\) outside a compact set of \(\mathbb{R}^+\).
Given a generating family \( F : \mathbb{R}^+ \times M \times \mathbb{R}^N \to \mathbb{R} \) for 
\( \psi(\bar{L}) \subset T^*(\mathbb{R}^+ \times M) \), and a function \( H : \mathbb{R}^+ \to \mathbb{R} \),

define the sheared difference function \( \Delta : M \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R} \) by:

\[
\Delta(t, x, e, \tilde{e}) = F(t, x, \tilde{e}) - F(t, x, e) + H(t).
\]
 Wrapped Generating Family Cohomology Groups

Given a generating family $F : \mathbb{R}^+ \times M \times \mathbb{R}^N \rightarrow \mathbb{R}$ for

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**KEY:** There is a non-degenerate critical submanifold with value 0
diffeomorphic to the compact $L$, and non-degenerate critical points
corresponding to the Reeb chords of $\Lambda_{\pm}$. 
Given a generating family \( F : \mathbb{R}^+ \times M \times \mathbb{R}^N \to \mathbb{R} \) for \( \psi(L) \subset T^*(\mathbb{R}^+ \times M) \), and a function \( H : \mathbb{R}^+ \to \mathbb{R} \),

define the **sheared difference function** \( \Delta : M \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R} \) by:

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WGH^*(F) := H^*(\Delta^\infty, \Delta^c).
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$$WGH^*(F) := H^*(\Delta^\infty, \Delta^\epsilon).$$

In fact, $WGH^*(F) \simeq H^*(L, \Lambda_{\pm}).$
$(\Delta^\infty, \Delta^\epsilon)$ as a Relative Mapping Cone

Classical Mapping Cone:

Given $f : X \to Y$, $C(f) := C(X) \cup Y / \sim$. Induced long exact sequence:

$$
\cdots \to H^k(Y) \xrightarrow{f^*} H^k(X) \to H^{k+1}(C(f)) \to H^{k+1}(Y) \xrightarrow{f^*} \cdots
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Relative Mapping Cone: Given a map $g : (X, A) \to (Y, B)$, there is a relative mapping cone $C(g)$ and a long exact sequence:

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\cdots \to H^k(Y, B) \xrightarrow{g^*} H^k(X, A) \to H^{k+1}(C(g)) \to H^{k+1}(Y, B) \xrightarrow{g^*} \cdots
$$
Classical Mapping Cone:

\[
\begin{array}{cccc}
& y & \leftarrow & x \\
\sigma & \cap_{\infty} & \cap_{\infty} & \cap_{\infty}
\end{array}
\]

Given \( f : X \to Y \), \( C(f) := C(X) \cup Y / \sim \). Induced long exact sequence:

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\]

\( (\Delta^\infty, \Delta^\epsilon) \) can be viewed as \( C(\psi) \) where \( \psi : \left(\delta_{\infty}, \delta_{\epsilon}\right) \to \left(\delta_{\infty}, \delta_{\epsilon}\right) \).

So, there is a **Cobordism Exact Sequence**!
Cobordism Long Exact Sequence

Theorem ([Sabloff-LT])

Given \((\Lambda_-, f_-) \prec_{(L, F)} (\Lambda_+, f_+)\), there is a long exact sequence:

\[
\cdots \to GH^k(f_-) \xrightarrow{\Psi^F} GH^k(f_+) \to H^{k+1}(L, \Lambda_+) \to \cdots.
\]

There is a similar story for linearized Legendrian contact homology;

[Ekholm, Ekholm-Honda-Kálmán, Golovko]
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- \(\Psi^F\) is non-trivial, natural, and behaves nicely under gluings of cobordisms.

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**Cobordism Long Exact Sequence**

**Theorem ([Sabloff-LT])**

Given \((\Lambda_-, f_-) \prec (L, F) (\Lambda_+, f_+)\), there is a long exact sequence:

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- \(\Psi_F\) is non-trivial, natural, and behaves nicely under gluings of cobordisms.
- If \(\Lambda_- = \emptyset\) (filling), then \(GH^k(f_+) \cong H^{k+1}(L, \Lambda_+).\)

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- \(\Psi_F\) is non-trivial, natural, and behaves nicely under gluings of cobordisms.
- If \(\Lambda_- = \emptyset\) (filling), then \(GH^k(f_+) \cong H^{k+1}(L, \Lambda_+)\).
- If \(L\) is a concordance, then \(GH^k(f_-) \cong GH^k(f_+)\).

There is a similar story for linearized Legendrian contact homology;

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Application: Obstructions to Lagrangian Cobordisms

\[ m(5_2)^2 \quad tb=1 \]
\[ P = \{ t^{-2} + t + t^2 \} \]
\[ \not\exists \quad L \]

\[ m(5_2)^1 \quad tb=1 \]
\[ P = \{ 2 + t \} \]
\[ \exists \quad L? \]
Application: Obstructions to Lagrangian Cobordisms

\[ P = \{ t^{-2} + t + t^2 \} \]

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\[ \exists L \]

\[ P = \{ 2 + t \} \]

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