Computation with Topological Defects

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Including joint work by
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Workshop on Parameterized Morse Theory
Banff International Research Station
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Introduction

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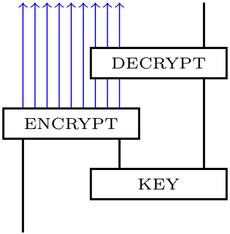

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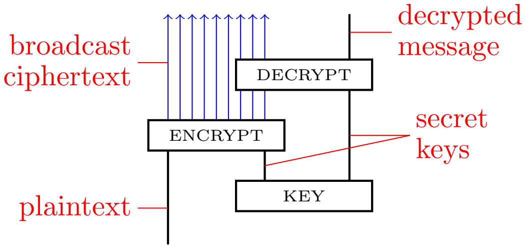

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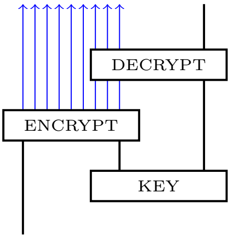

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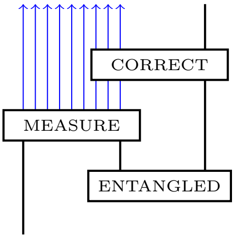

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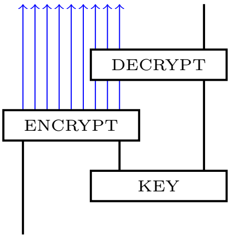

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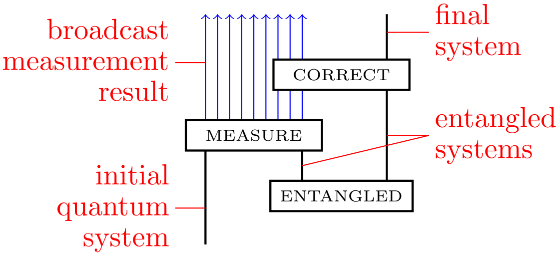

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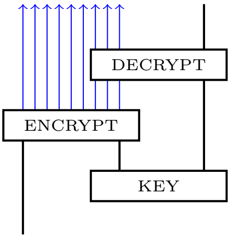

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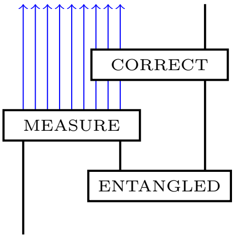

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Main idea. These procedures, and others, can be exactly characterized in terms of *defects* between 1d and 2d topological cobordisms.

Describing extended systems

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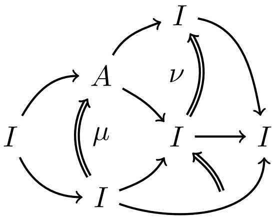

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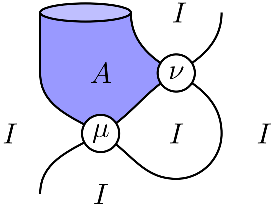

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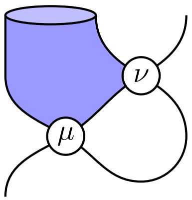

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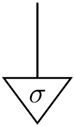

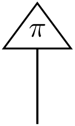

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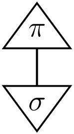

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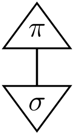

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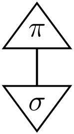

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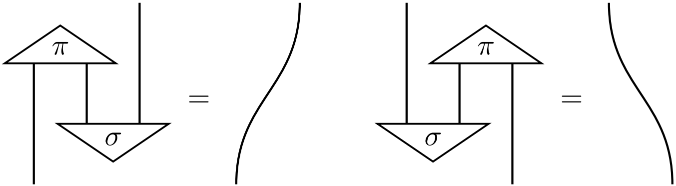
- If S is a set and σ, π are spans, this is an integer.
- If S is a Hilbert space and σ, π are linear maps, this is a complex number giving a probability amplitude.

Strings and correlation

A state σ of $S \times S'$ is *perfectly correlated* if there is a property π satisfying the following conditions:



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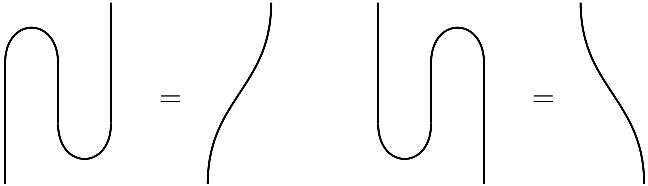

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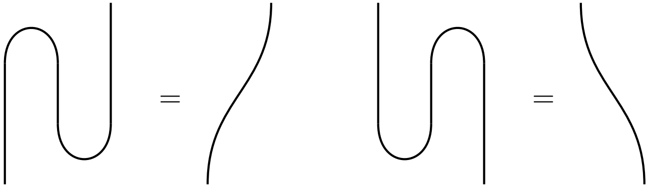

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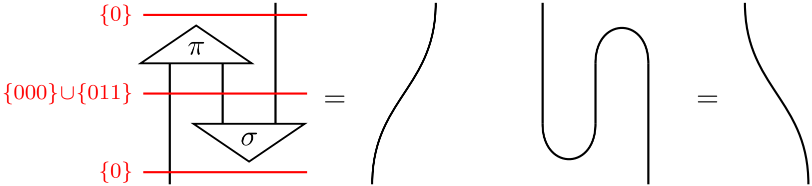

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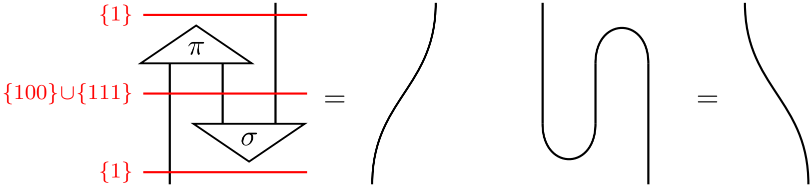

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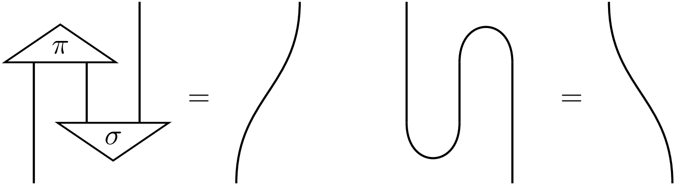

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- **Sets and spans.** If $S = S' = \{0, 1\}$, then
$$\sigma = \{00\} \cup \{11\} \qquad \pi = \{00\} \cup \{11\}$$
- **Hilbert spaces and linear maps.** If $S = S' = \mathbb{C}^2$, then
$$\sigma = |00\rangle + |11\rangle \qquad \pi = \langle 00| + \langle 11|$$
This is *quantum entanglement*.

Surfaces and logic

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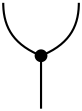
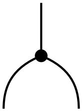
In particular, we want to be able to copy, compare and delete information. We can represent these processes graphically:



copy



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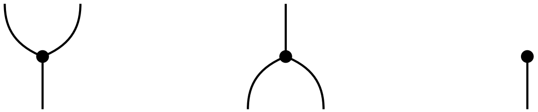


copy compare delete

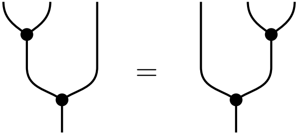
We might imagine these acting on some set S as spans:

$$\text{copy} : \{s\} \mapsto \{s, s\}$$
$$\text{compare} : \{s, t\} \mapsto \delta_{s,t}\{s\}$$
$$\text{delete} : \{s\} \mapsto 1$$

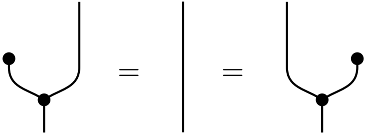
How can we axiomatize these operations?

Surfaces and logic

These copying, comparing and deleting processes satisfy some interesting equations:



Associativity



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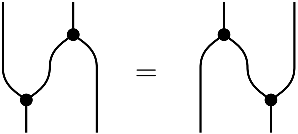
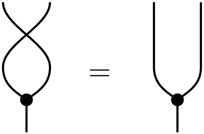


Frobenius law



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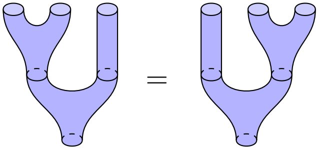

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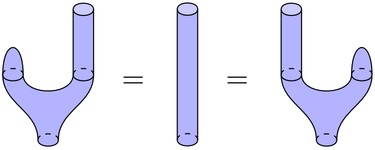

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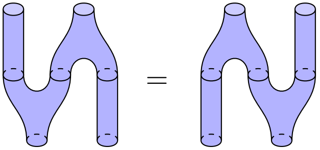

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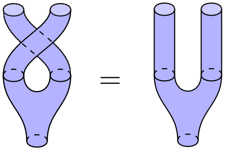

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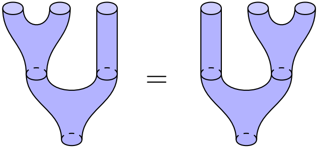

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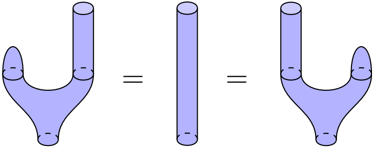

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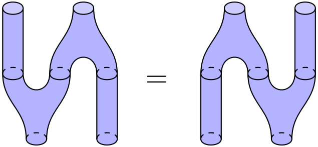

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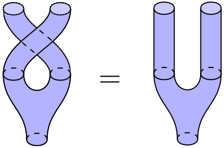

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So we need exactly a **2d topological field theory**.

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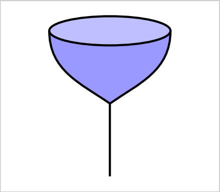

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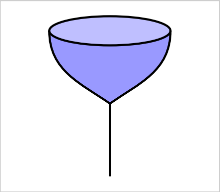

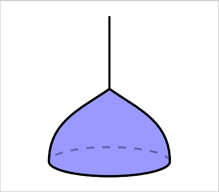

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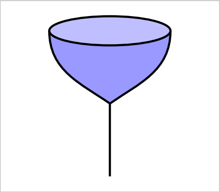
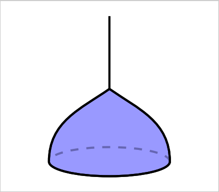
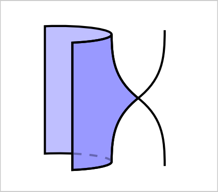

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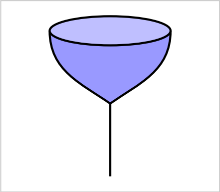

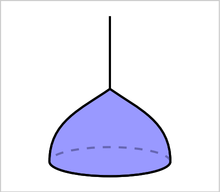

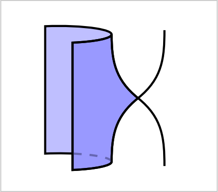

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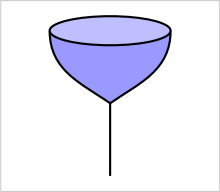

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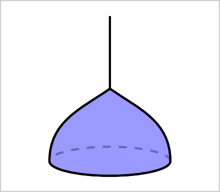
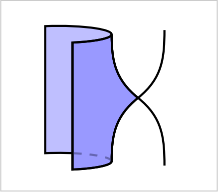


Retrieval



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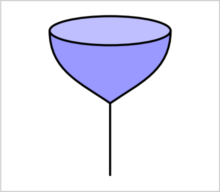

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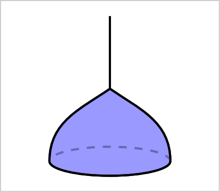

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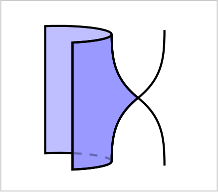

Controlled operation

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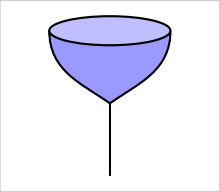

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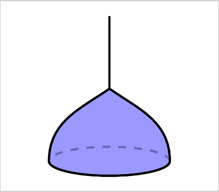

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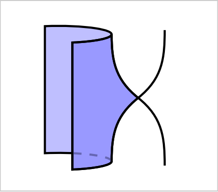

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We require these to be invertible, because *all* processes in physics and computer science are (arguably) reversible at a fundamental level.

Models

How do these abstract processes relate to real computation?

- We choose symmetric monoidal 2-categories in which to represent our defect structures.

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- Computational tasks can be described abstractly as *equations* involving our defect structures.
- Performing a task concretely means finding a solution to the equation in the appropriate 2-category.

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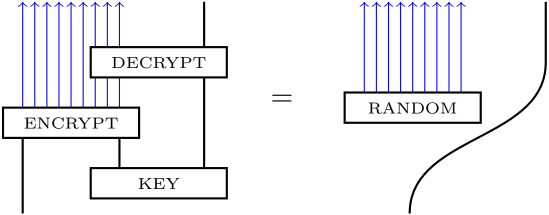
Conjecture. The 2-category **2Gpd** is symmetric monoidal.

Encrypted communication

This heuristic equation represents encrypted communication:



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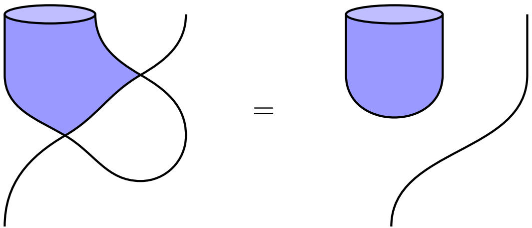

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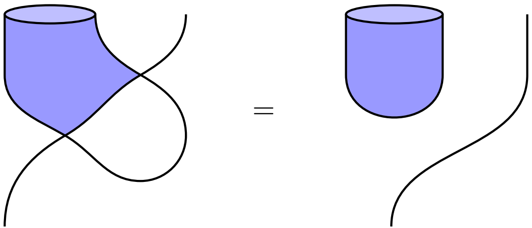

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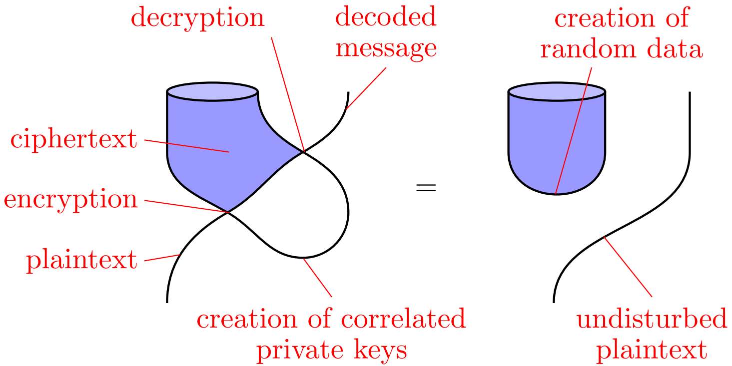

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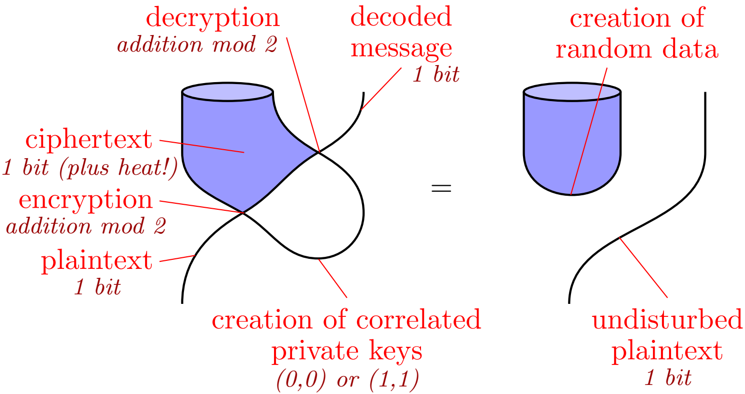

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This is equivalent to the 2-category **Alg** of algebras, bimodules and bimodule homomorphisms.

It is a standard target 2-category in higher representation theory.

Quantum teleportation

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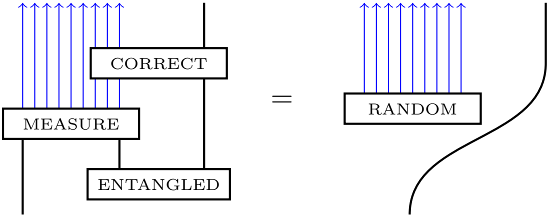

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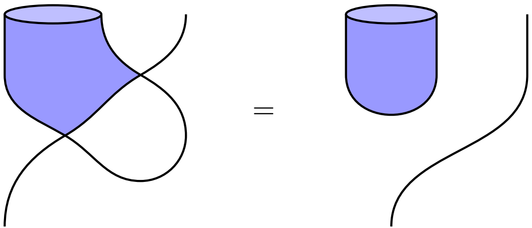

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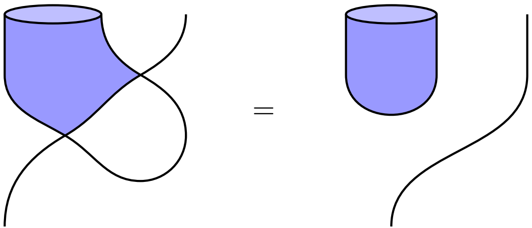

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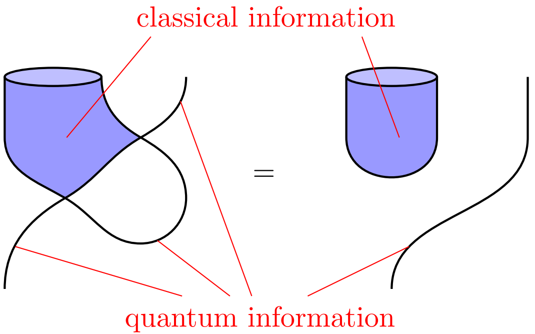

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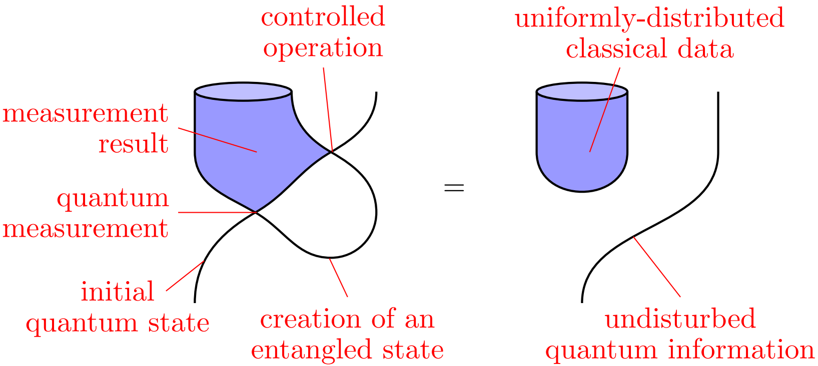

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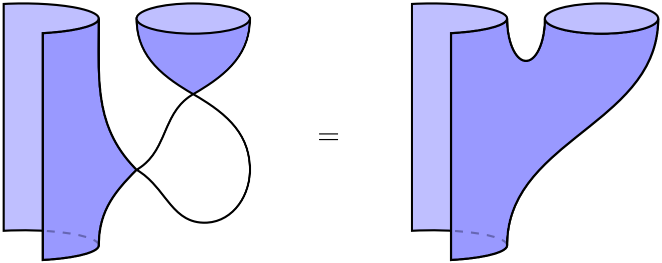

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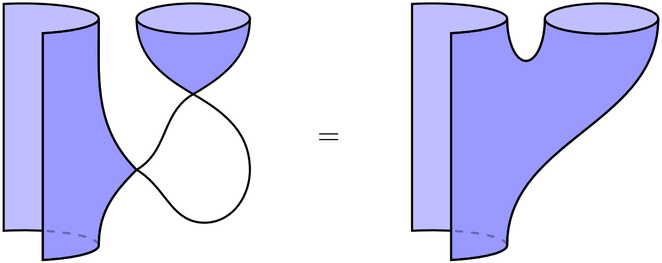

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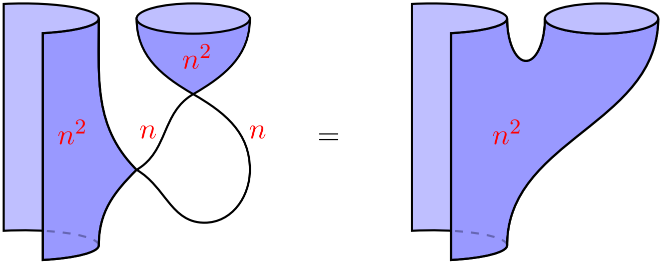

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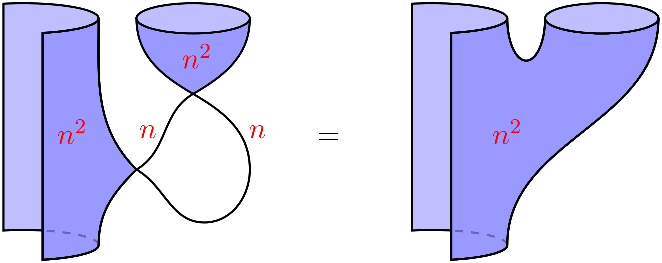

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Theorem. Solutions of this equation in **2Hilb** correspond exactly to implementations of quantum dense coding.

The equation also has *classical* solutions, which seem to be new.

Quantization

There is a quantization 2-functor

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Theorem. The 2-functor Q maps encrypted communication into quantum teleportation.

This gives the first functorial *combinatorial model* for quantum teleportation.

Further directions

- Some mathematical details need to be checked:
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- Connects low-dimensional topology to foundational issues in computer science:
  - What are the conditional operations in thermodynamics?
  - Is measurement unitary in quantum physics?

Thank you!