Vojta's Conjectures

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1 Overview of the Field

In his seminal work [16], Paul Vojta opened the door to a vast network of correspondences between Nevanlinna Theory and Diophantine geometry, culminating in a set of sweeping conjectures that aimed at unifying a huge swath of number theory and arithmetic geometry. Vojta's work provided a framework and context for a variety of powerful results in number theory, a new proof of an old conjecture, and illuminated the path forward to new results in virtually every area of arithmetic geometry.

The two fields united by Vojta's conjectures have a long history, although Diophantine geometry, by its nature, is much older, dating back – in a sense at least – to ancient Greece. The modern part of the story therefore cannot have a clear beginning, but a convenient starting point is the chain of results begun by Liouville in his 1844 paper ([7]).

Theorem 1 (Liouville 1844)**.** *Let α be a real algebraic number, and let $d = [\mathbb{Q}(\alpha) : \mathbb{Q}]$. There are only finitely many rational numbers p/q such that:*

$$\left|\alpha - \frac{p}{q}\right| \leq \frac{1}{q^d}$$

Liouville's central observation was that a rational number cannot get too close to an algebraic one. Liouville applied his theorem to construct the first concrete example of a transcendental number, by finding a real number that, for any d, could be approximated better than his theorem prescribes for an algebraic number. This kind of clever application of Diophantine geometry to transcendence theory continues to this day – in work of Adamczewski and Bugeaud (see [1]), for example.

Liouville's result, revolutionary though it was, was not best possible for $d \geq 3$. (It is sharp for $d = 1$ and $d = 2$.) In particular, the exponent d on the right hand side is much larger than necessary for large d. A famous chain of results followed to improve this, starting with Axel Thue in 1909 ([15]). He reduced the exponent from d to $d/2 + 1 + \epsilon$. Carl Siegel reduced the exponent further to $2\sqrt{d}$, in 1921 ([14]), and in 1947 Freeman Dyson reduced it still further to approximately $\sqrt{2d}$ ([5]).

However, none of these results were optimal, even though they all had profound applications to arithmetic geometry. The capstone theorem of this line of research was the 1955 masterwork of Klaus Roth ([11]).

Theorem 2 (Roth 1955). *Let α be a real algebraic number. There are only finitely many rational numbers p/q such that:*

$$\left|\alpha - \frac{p}{q}\right| \leq \frac{1}{q^{2+\epsilon}}$$

The value of $2+\epsilon$ for the exponent is known to be the best possible because of the continued fraction expansion of α, which gives an infinite number of rational approximations of α with exponent 2. So Roth's Theorem, it would seem, is the end of the story.

Except, of course, that whenever a mathematical problem is solved, several more problems arise. For example, the left side of the inequality in Roth's Theorem is phrased in terms of the usual archimedean distance. What happens if one uses a p-adic distance instead? (The exponent of $2+\epsilon$ is unchanged – see [10].) Or a finite product of distances instead of just one? (Same exponent again, essentially – see [10] again.)

A much deeper generalization of Roth's Theorem was found in 1970 by Wolfgang Schmidt (see [12]). Roth's Theorem is essentially a one-dimensional statement, about approximations on the line. Schmidt's theorem vastly extended the scope, to approximations in arbitrary finite dimensions. It was generalized in [13] to the following.

Theorem 3 (Subspace Theorem). *Let k be a number field with ring of integers $\mathcal{O}_k$, and let $n \geq 1$ be a positive integer. Let S be a finite set of places of k. For each $v \in S$ and each $i \in \{0, \ldots, n\}$, let $L_{v,i}$ be a linear form in $n+1$ variables with algebraic coefficients, and assume that for each v, the forms $L_{v,0}, \ldots, L_{v,n}$ are linearly independent.*

Choose $\epsilon > 0$, and let Q be the set of all $s \in \mathcal{O}_k^{n+1}$ satisfying

$$\prod_{v\in S}\prod_{i=0}^{n} |L_{v,i}(s)|_v < \left(\max_{v|\infty,\, i} |s_i|_v\right)^{-\epsilon}$$

where $|\cdot|_v$ denotes the absolute value associated to v and s_i denotes the ith coordinate of s.

Then Q is contained in a finite union of hyperplanes of k^{n+1}.

This was the state of the art in Diophantine geometry when Paul Vojta formulated his conjectures. The other half of the story is in Nevanlinna theory, which was the source of the key inspiration.

Nevanlinna theory was inspired by a result of Hadamard in 1896, probably, although he did not publish a proof of it. Hadamard noted that the logarithm of the maximum modulus of an entire function inside the circle $|z| = r$ grows at least as fast as the number of zeroes of f inside the circle, as a function of r. The key breakthrough was made by Nevanlinna, however, when he replaced the notion of maximum modulus with a counting function, enabling him to deal with general meromorphic functions instead of entire functions. (See [9] for details.) To state Nevanlinna's results, we need some additional terminology and notation.

- For any real valued function A, define $A^+(x) = \max\{A(x), 0\}$.
- Let : $\mathbb{C} \to \mathbb{C}$ be a meromorphic function.
- For any $a \in \mathbb{C}$, define the proximity function of f at a to be $m(a,r) = \int_0^{2\pi} \log^+ \left|\frac{1}{f(re^{i\theta})-a}\right| \frac{d\theta}{2\pi}$.
- Define the proximity function of f at infinity to be $m(\infty, r) = \int_0^{2\pi} \log^+ \left|f(re^{i\theta})\right| \frac{d\theta}{2\pi}$.
- For any complex number a and positive real number r, define $n(a,r)$ to be the number of zeroes of $f-a$ inside the circle $|z| = r$.
- Define $n(\infty, r)$ to be the number of poles of f inside $|z| = r$.
- For any complex number a, define the counting function of f at a to be the integral $N(a,r) = \int_0^r n(a,s)\frac{ds}{s}$.
- Similarly, define the counting function at infinity to be $N(\infty, r) = \int_0^r n(\infty, s)\frac{ds}{s}$.
- Define the characteristic function of f to be $T(r) = \int_0^{2\pi} \log^+ \left|f(re^{i\theta})\right| \frac{d\theta}{2\pi} + N(\infty, r)$.

- Define $n_1(r)$ to be the number of ramification points of f in the disk $|z| < r$.
- If 0 is not a ramification points of f, define $N_1(r) = \int_0^r n_1(r) \frac{dr}{r}$.

Nevanlinna's First Main Theorem states:

Theorem 4 (First Main Theorem)**.** *For any meromorphic function $f\colon \mathbb{C} \to \mathbb{C}$, and any complex number a, $N(a,r) + m(a,r) = T(r) + O(1)$.*

Nevanlinna's much deeper Second Main Theorem is as follows.

Theorem 5 (Second Main Theorem)**.** *For any distinct complex numbers $a_1, \ldots, a_n$, the following inequality holds outside a set of bounded measure:*

$$\sum_{i=1}^{n} m(a_i, r) \leq 2T(r) - N_1(r) + O(\log rT(r))$$

The Second Main Theorem deals with an arbitrary finite set of complex numbers rather than just one, and involves the derivative of f in a highly nontrivial way through the term $N_1(r)$. The third member of Nevanlinna's trio of foundational theorems is the defect relation.

Theorem 6 (Defect relation)**.** *For any non-constant meromorphic function $f\colon \mathbb{C} \to \mathbb{C}$, we have the following inequality:*

$$\sum_{a \in \mathbb{C}} \mathit{liminf}_{r \to \infty} \frac{m(a,r)}{T(r)} \leq 2$$

These notions have been generalised to higher dimensions – and arbitrary smooth varieties – by various authors. The higher dimensions have appeared in both the domain and the codomain of the meromorphic maps, but we will concentrate on the codomain in this report. In order to do this, we must define the basic objects in higher dimensions.

- Let V be a smooth projective variety defined over $\mathbb{C}$.
- Let $f\colon \mathbb{C} \to V$ be a non-constant meromorphic function.
- Let D be a normal crossings divisor on V.
- A Weil function for D is a function $\lambda_D\colon (V - \mathrm{Supp}(D))(\mathbb{C}) \to \mathbb{R}$ such that if f is a local defining function for D, then $\lambda_D + \log|f|$ is continuous.
- The proximity function for D is $m_f(D,r) = \int_0^{2\pi} \lambda(f(re^{i\theta})) \frac{d\theta}{2\pi}$
- The counting function for D is $N_f(D,r) = \sum_{0<|z|<r} \mathrm{ord}_z f^*D \cdot \log \frac{r}{|z|} + \mathrm{ord}_0 f^*D \cdot \log r$
- The characteristic function (or height function) for D is $T_{D,f}(r) = m_f(D,r) + N_f(D,r)$.

With these definitions, the First Main Theorem becomes a tautology, although the fact that T depends only on the linear equivalence class of D rather than the particular divisor is also well known (and not immediate) – see Proposition 11.6 of [17] for a proof.

The generalised Second Main Theorem, however, is still unknown. It is apparently originally due to Griffiths, but [17] contains a thorough discussion. For the statement used below (taken from [17]), we assume that K is the canonical line bundle on V, and A is an ample line bundle on V.

Conjecture 7. *(a) The inequality*

$$m_f(D,r) + T_{K,f}(r) \leq O(\log^+ T_{A,f}(r)) + o(\log r)$$

holds for all holomorphic curves $f\colon \mathbb{C} \to V$ with Zariski dense image, except for r lying in a set of finite Lebesgue measure. (b) For any $\epsilon > 0$, there is a proper Zariski closed subset Z of X (depending only on X, D, A, and ϵ) such that the inequality

$$m_f(D,r) + T_{K,f}(r) \leq \epsilon T_{A,f}(r) + C$$

holds (except for r lying in a set of finite Lebesgue measure) for all nonconstant holomorphic curves $f\colon \mathbb{C} \to V$ whose image is not contained in Z, and for all $C \in \mathbb{R}$.

Vojta's critical insight was to realise that there were deep connections to be made between these various theorems and the objects they describe. Vojta's dictionary between the two fields contains the following correspondences:

- A meromorphic function $f\colon \mathbb{C} \to \mathbb{C}$ corresponds to an infinite set of elements of a number field k.
- The characteristic function $T(r)$ corresponds to the height function $h(b) = \frac{1}{[k:\mathbb{Q}]} \sum_v \log^+ \|b\|$.
- Let S be a finite set of places of k. Then there are analogues of m and N for the Diophantine case:
  $m(a, b) = \frac{1}{[k:\mathbb{Q}]} \sum_{v \in S} \log^+ \|b\|$
  $N(a, b) = \frac{1}{[k:\mathbb{Q}]} \sum_{v \notin S} \log^+ \|b\|$

The correspondence extends to theorems as well. Roth's Theorem corresponds to the Defect Relation in Nevanlinna theory, using the correspondences described above. The First Main Theorem in Nevanlinna theory becomes a simple statement about the height of an algebraic number, following immediately from the definition. The Second Main Theorem in Nevanlinna theory, however, becomes something much, much deeper, which Vojta called his Main Conjecture.

To state the Main Conjecture, we need to define some notation.

- Let V be a smooth projective variety defined over k.
- Let D be a normal crossings divisor on V, also defined over k.
- Let K be the canonical divisor class on V.
- Let A be a big divisor class on V.
- Let ϵ be any positive real number.
- Let S be a finite set of places of k.

Conjecture 8 (Vojta's Main Conjecture)**.** *There is a proper Zariski closed subset Z of V, depending on V, D, k, A, ϵ, and S, such that for all $P \in V(k)$ with $P \notin Z$, we have*

$$m(D, P) + h_K(P) \leq \epsilon h_A(P) + O(1)$$

where the implied constant in the $O(1)$ does not depend on P.

This conjecture is a generalization of the Subspace Theorem, and therefore of all the other theorems in Diophantine geometry described so far. It also generalises Siegel's Theorem for curves, Faltings' Theorem (originally the Mordell Conjecture), and the Bombieri-Lang conjecture for varieties of general type. Vojta's Main Conjecture is known to imply many of the major conjectures in modern Diophantine geometry, including the abc Conjecture, Hall's Conjecture (see [16] for details on these two), and the Batyrev-Manin Conjecture for varieties of non-negative Kodaira dimension (see [8]). It also has deep consequences in transcendence theory and on the distribution of integral and rational points on algebraic varieties.

2 Recent Developments and Open Problems

Boris Adamczewski and Yann Bugeaud reignited interest in applying Vojta-like ideas to transcendence theory with their paper [1], which uses the Subspace Theorem to the complexity of the b-ary expansions of irrational numbers, and in particular proving the transcendence of irrational automatic numbers. To explain their results more fully, we need some definitions.

Let $b \geq 2$ be an integer, and let $x = 0.a_1a_2a_3\ldots$ be a real number, expressed in base b. That is, we have $x = \sum_i a_i b^{-i}$, with $a_i \in \{0, \ldots, b-1\}$. The complexity function of x in base b is a function $\rho_x\colon \mathbb{Z}_{\geq 1} \to \mathbb{Z}$, where $\rho_x(n)$ is the number of distinct strings of digits $d_1 \ldots d_n$ of length n that appear somewhere in the b-ary expansion of x. That is, a string $d_1 \ldots d_n$ contributes to $\rho_x(n)$ if and only if there is some integer j such that $d_i = a_{i+j}$ for all $i \in \{1, \ldots, n\}$. To convert this measure of complexity into a number, one can estimate the growth of $\rho_x(n)$ as a function of n. Rational numbers will have $\rho_x(n) = O(1)$, because their b-ary expansions are periodic. Adamczewski and Bugeaud, by contrast, showed that irrational algebraic numbers have complexity functions that grow faster than any linear functions: (see [1])

Theorem 9. *Let $\alpha \in (0,1)$ be an irrational algebraic number, and let $b \geq 2$ be an integer. The b-ary complexity function $\rho_\alpha(n)$ of α satisfies*

$$\lim_{n\to\infty} \frac{\rho_\alpha(n)}{n} = \infty$$

This means that any real number whose complexity function grows linearly (or more slowly) must either be rational or transcendental. In particular, automatic numbers are well known to have complexity functions that are $O(n)$, and are therefore either rational or transcendental. In particular, the irrational automatic numbers are all transcendental.

This theorem was derived from an earlier result of Adamczewski, Bugeaud, and Luca ([2]).

Theorem 10. *Let $\alpha \in (0,1)$ be a real number, and let $b \geq 2$ be a positive integer. Assume that for every positive integer N and every $\epsilon > 0$ there are two identical disjoint sequences of digits of length at least ϵN in the first N digits of the b-ary expansion of α. Then $alpha$ is either rational or transcendental.*

Pietro Corvaja and Umberto Zannier, together with a variety of other authors, discovered an ingenious new method of applying the Subspace Theorem to the question of determining the distribution of integral points on algebraic varieties. The following theorem appears in [4].

Theorem 11. *Let V be a smooth projective surface defined over a number field k, and let $U \subset \mathbb{A}^n$ be a nonempty affine subset of V. Let $D_1, \ldots, D_r$ be effective (but not necessarily irreducible) divisors on V supported on $V - U$. Assume that no two of the D_i have a common component, and that no three of the D_i have a common point. Assume that for some integers $a_1, \ldots, a_r, n$, $D = a_1D_1 + \ldots + a_rD_r$, we have*

$$\frac{\sum_{k=0}^{\infty} h^0(nD - kD_i)}{nh^0(nD)} > a_i$$

for all i. Then for any finite set of places S of k, the set of S-integral points on U is not Zariski dense.

Many authors have followed up on these ideas, and have improved the results. A particularly impressive example was obtained independently by Autissier ([3]) and Levin ([6]).

Theorem 12. *Let V be a smooth projective surface defined over a number field k, and let $U \subset \mathbb{A}^n$ be a nonempty affine subset of V. Let $D_1, \ldots, D_r$ be effective (but not necessarily irreducible) divisors on V supported on $V - U$. Assume that no two of the D_i have a common component, and that no three of the D_i have a common point. If $r \geq 4$, then for any finite set S of places of k, the S-integral points of U are not Zariski dense.*

In a still different direction, David McKinnon ([8]) proved that Vojta's conjectures imply the Batyrev-Manin conjectures for an arbitrary smooth projective variety of non-negative Kodaira dimension. In particular, he proves this implication for a $K3$ surface, which has a particularly interesting role in the conjectures.

To state the main theorem, we must first describe the Batyrev-Manin conjecture for a $K3$ surface. It gives a conjectural description of the distribution of rational points on the surface, in terms of the height density. In particular, let X be a $K3$ surface defined over a number field k. For any real number B, and for any subset U of X, let $N_U(B)$ be the number of k-rational points of U of height at most B. The Batyrev-Manin Conjecture states that

Conjecture 13. *Let $\epsilon > 0$ be any positive real number. Then there is a nonempty Zariski open subset $U(\epsilon) \subset X$ such that*

$$N_{U(\epsilon)}(B) = O(B^\epsilon)$$

In this conjecture, we are using the multiplicative height rather than the logarithmic one used earlier. Note also that the conjecture implies that the surface X is presented as a subvariety of projective space, in order to define the height functions. This corresponds to a choice of (very) ample divisor on X, and this dependence is often made explicit.

It is known that $K3$ surfaces contain rational curves, and it is widely believed that the number of such curves is always infinite. Certainly there are examples of $K3$ surfaces that contain an infinite number of rational curves, and in many of these there are infinitely many such curves defined over k. In these cases, the surface X will contain curves which violate the Batyrev-Manin inequality, and so these curves must be in the complement of the set $U(\epsilon)$ for small enough ϵ. These curves will turn out to be related to the exceptional sets Z that appear in Vojta's Conjecture.

In [8], McKinnon proves the following theorem.

Theorem 14. *Let X be a smooth, projective surface of nonnegative Kodaira dimension, defined over a number field k. Assume that Vojta's Main Conjecture for general cycles is true for any variety birational to a subvariety of $X \times X$. Then for every $\epsilon > 0$ and ample divisor L on X, there is a nonempty Zariski open subset $U(\epsilon) \subset X$ such that*

$$N_{U(\epsilon),L}(B) = O(B^{\epsilon}).$$

In particular, Vojta's Main Conjecture implies the Batyrev-Manin Conjecture for $K3$ surfaces.

Vojta's Main Conjecture for general cycles is a natural generalization of Vojta's Main Conjecture, with the slightly relaxed hypothesis that the divisor D might have codimension larger than one. The normal crossings condition is generalised to demanding that the cycle D be contained in a divisor with normal crossings. This generalization of Vojta's Main Conjecture is an immediate consequence of the original. See [8] for details.

3 Presentation Highlights

The workshop was filled with interesting presentations. The workshop was inaugurated by three introductory talks by Paul Vojta, Frédéric Campana, and Aaron Levin, who set the stage for the week to come. Vojta's talk laid out the conceptual framework for his conjectures, and outlined recent developments in broad terms. Campana reported on the progress of research in Vojta's conjectures as they apply to certain kinds of fibrations, and in particular on the effect of non-reduced components of singular fibres on the distribution of rational points. After lunch, Levin gave an excellent overview of the recent impressive results on integral points deriving from clever applications of the Subspace Theorem by Corvaja, Levin, and Zannier.

Ekaterina Amerik described joint work with Frédéric Campana on the characteristic foliation on smooth divisors in holomorphically symplectic manifolds. Let D be a smooth divisor in a holomorphically symplectic manifold X. Then D carries a rank-one foliation obtained as a kernel of the restriction of the symplectic form to D. It is called the characteristic foliation. Hwang and Viehweg have proved that when D is of general type, the foliation cannot be algebraic (unless in the trivial case when X is a surface). On the other hand, it is easy to see that the foliation is algebraic when D is uniruled. She explained what happens in the case when D is not uniruled. In particular, if X is an irreducible holomorphic symplectic manifold (that is, simply connected and such that the holomorphic symplectic form is unique up to a constant), she and Campana proved that the characteristic foliation is never algebraic in this case. The main new ingredient for their results is the observation that the canonical bundle of the orbifold base of the family of leaves must be torsion. This implies, in particular, the isotriviality of the family of leaves.

Yann Bugeaud spoke about connections of Vojta's conjectures to transcendence theory. A few years after Roth's fundamental breakthrough, Cugiani proved a theorem with the same conclusion, except with the ϵ replaced by a function $q \mapsto \epsilon(q)$ that decreases very slowly to zero, provided that the sequence of rational solutions to $|\xi - p/q| < q^{-2-\epsilon(q)}$ is sufficiently dense, in a suitable sense. He described an alternative, and much simpler, proof of Cugiani's Theorem and extended it to simultaneous approximation.

William Cherry gave an excellent survey of the non-Archimedean analog of Nevanlinna's theory of value distribution initiated by Ha Huy Khoai, focussing on similarities to and differences from the complex case. He also discussed the work of An, Levin, and Wang which introduces a Vojta like analogy between non-Archimedean analytic curves and integral points over the rationals and imaginary quadratic number fields. Finally, he also surveyed criteria for degeneracy of non-Archimedean analytic maps from the affine line into algebraic varieties and highlight an open problem whose solution might shed some light on the Green-Griffiths conjecture in complex geometry.

Pietro Corvaja, one of the major contributors to the recent impressive theorems on integral points, gave a presentation about joint work with Thomas Tucker, Vijay Sookdeo, and Umberto Zannier on integral points in orbits for a two-dimensional dynamical system. Let K be a number field, let $f : \mathbb{P}_1 \dashrightarrow \mathbb{P}_1$ be a nonconstant rational map of degree greater than 1, let S be a finite set of places of K, and suppose that $u, w \in \mathbb{P}_1(K)$ are not preperiodic under f. We prove that the set of $(m, n) \in \mathbb{N}^2$ such that $f^{\circ m}(u)$ is S-integral relative to $f^{\circ n}(w)$ is finite and effectively computable. This may be thought of as a two-parameter analog of a result of Silverman on integral points in orbits of rational maps. This issue can be translated in terms of integral points on an open subset of $\mathbb{P}_1^2$; then one can apply a modern version of the method of Runge, after increasing the number of components at infinity by iterating the rational map. Alternatively, an ineffective result comes from a well-known theorem of Vojta.

A very deep and comprehensive presentation came from Jan-Hendrik Evertse, surveying recent results and open problems relating to the Subspace Theorem. Let $L_i = \sum_{i=1}^{n} \alpha_{ij} X_j$ $(i = 1 \ldots n)$ be n linearly independent linear forms with algebraic coefficients (from $\mathbb{C}$), and let $c_1 \ldots c_n$ be reals. Denote by $\|\mathbf{x}\|$ the maximum norm of $\mathbf{x} \in \mathbb{Z}^n$. Consider the system of inequalities

$$|L_i(\mathbf{x})| \leq \|\mathbf{x}\|^{c_i} \ (i = 1 \ldots n) \quad \text{in } \mathbf{x} \in \mathbb{Z}^n. \tag{1}$$

The following result is equivalent to Schmidt's Subspace Theorem from 1972:

Assume that $c_1 + \cdots + c_n < 0$. Then the set of solutions of (1) *is contained in finitely many proper linear subspaces of $\mathbb{Q}^n$.*

The proofs of the Subspace Theorem given so far are ineffective, in that they do not allow to compute the subspaces containing the solutions. But work of Vojta (1989), Schmidt (1993) and Faltings and Wüstholz (1994) implies that Schmidt's Subspace Theorem can be refined as follows:

Assumptions being as above, there is an effectively computable, proper linear subspace T^{exc} of $\mathbb{Q}^n$ such that (1) *has only finitely many solutions outside T^{exc}. Moreover, T^{exc} can be chosen from a finite collection independent of $c_1 \ldots c_n$.*

Here, the method of proof does not allow to compute the solutions outside T^{exc}. But one can prove the following 'semi-effective' result. Assume that the coefficients of $L_1 \ldots L_n$ have heights at most H, and that they generate a number field K of degree D. Further, let $c_1 + \cdots + c_n \leq -\delta$ with $0 < \delta < 1$, and $\max(c_1 \ldots c_n) = 1$. Then one can show that for the solutions $\mathbf{x} \in \mathbb{Z}^n$ of (1) outside T^{exc} one has

$$\|\mathbf{x}\| \leq \max\left(B^{\text{ineff}}(n, K, \delta),\ H^{c^{\text{eff}}(n,D,\delta)}\right),$$

where c^{eff} is effectively computable, B^{ineff} not effectively computable from the method of proof, and both constants depend only on the parameters between the parentheses.

Evertse posed as an open problem to replace the above bound by one in which B^{ineff} depends on D instead of K. This would have various interesting consequences, of which he mentioned a few during the talk.

Kálmán Győry gave a survey of some recent results, partly obtained jointly with A. Berczes and J.-H. Evertse on various classes of Diophantine equations, including unit equations and some of their generalizations, Thue equations, superelliptic equations with unknown exponents, discriminant equations and their applications. Their results make effective several ineffective results of Lang and others, and generalize many earlier results established over number fields.

Gordon Heier discussed joint work with Aaron Levin regarding a generalization of the Schmidt subspace theorem (and Cartan's Second Main Theorem) in the setting of not numerically equivalent divisors. His presentation fit beautifully into the framework of the workshop, reinforcing the link between Nevanlinna theory (or, more generally, value distribution theory) and Diophantine geometry.

Khoa Nguyen presented joint research with Thomas Tucker, Dragos Ghioca, and Chad Gratton about primitive and doubly primitive divisors in dynamical sequences. Let K be a number field or a function field of characteristic 0, let $\varphi(z) \in K(z)$ having degree at least 2 and let $\alpha \in K$ such that the orbit $\{\varphi^n(\alpha)\}_{n \geq 0}$ is infinite. Consider the question: (A) except trivial counter-examples, is it true that for all sufficiently large n, the element $\varphi^n(\alpha)$ has a prime divisor $\mathfrak{p}$ that is not a divisor of $\varphi^k(\alpha)$ for every $k < n$. Ingram and Silverman are the first to consider this question in such generality. They even go further and ask: (B) except trivial counter-examples, is it true that for all $m \geq 0$ and $n > 0$ such that $m + n$ is sufficiently large, the element $\varphi^{m+n}(\alpha) - \varphi^m(\alpha)$ has a prime divisor $\mathfrak{p}$ that is not a divisor of any $\varphi^{M+N}(\alpha) - \varphi^M(\alpha)$ for $M < n$ or $N < n$. Later on, Faber and Granville modify question (B) somewhat and provide certain evidence towards it.

Nguyen explained how the abc Conjecture (well known to be a consequence of Vojta's conjectures) implies that both questions have an affirmative answer. In the function field case their result is unconditional; when using a deep result of Yamanoi (previously conjectured by Vojta), he showed that $\mathfrak{p}$ appears with multiplicity 1.

In his presentation on the radical of polynomial values, Hector Pasten pointed out that it is known that the abc conjecture gives a good lower bound for the radical of $F(n)$ in terms of n, for any fixed polynomial F without repeated factors. He further explained related results where one is interested in lower bounds in terms of F rather than n. This leads to new applications of the abc conjecture, such as counting square free values of polynomials at prime arguments (which also uses results of Green, Tao and Ziegler about arithmetic progressions of primes), and some consequences in undecidability questions (related to Hilbert's tenth problem).

Min Ru's presentation was about quantitative geometric and arithmetic results for complements of divisors. He introduced, for an effective divisor D on a smooth projective variety X, the notion of Nevanlinna constant. He then explained his proof of a quantitative result, in terms of the Nevanlinna constant of D, which extends the Subspace Theorem to D in X. He also derived the counterpart in Nevanlinna theory. The result recovers the recent important results in this direction, including the results of Corvaja-Zannier, Evertse-Ferretti, Levin, Ru etc., as well as deriving new results. The notion of "Nevanlinna constant of D" gives a unified description of the quantitative geometric and arithmetic properties of (X, D).

Amos Turchet presented a proof of the function field version of Lang-Vojta Conjecture on algebraic hyperbolicity for complements of very generic quartics in the projective plane with at most normal crossing singularities. The proof relies on a deformation argument applied to the known case for three components divisors (proved by Corvaja and Zannier) and uses a reformulation of the problem via moduli spaces of logarithmic stable maps as introduced by Q. Chen and Abramovich.

Paul Vojta himself gave a second talk, more focussed on a specific area of research, namely toric geometry and Dyson's lemma. In 1989, he proved a Dyson lemma for products of two smooth projective curves of arbitrary genus. In 1995, M. Nakamaye extended this to a result for a product of an arbitrary number of smooth projective curves of arbitrary genus, in a formulation involving an additional "perturbation divisor." In 1998, he also found an example in which a hoped-for Dyson lemma is false without such a perturbation divisor. This talk will present work in progress on eliminating the perturbation divisor by using a different definition of "volume" at the points under consideration. The proof involves toric and toroidal geometry, and this is reflected in the statement as well.

Felipe Voloch followed up his lively and heavily-debated contribution to the open problem session (described in the next section) by discussing some results and speculations about the number of rational points on curves of genus bigger than one over global fields. In particular, he described the result obtained jointly with R. Concei and D. Ulmer, where he showed that the number of rational points on (non-isotrivial) curves of fixed genus over a fixed function field can be arbitrarily large.

As grand finale to the workshop, Julie Tzu-Yueh Wang proved the rank one case of Skolems Conjecture on the exponential local-global principle for algebraic functions, and discussed its analog for meromorphic functions.

4 Scientific Progress Made

In addition to the various presentations, there were a great number of research collaborations that were initiated or continued at the workshop. Ekaterina Amerik and Frédéric Campana, for example, were working tirelessly on their joint project described by Amerik in her talk. McKinnon and Yongqiang Zhao also made substantial progress in their research collaboration, and in general the BIRS lounge and reading room were seldom empty during the evening, filled with workshop participants.

The most palpable evidence of scientific progress at the meeting, however, was made at the open problem session on Monday afternoon. Participants spent roughly an hour proposing interesting open questions to attack, and suggested ways of approaching their colleagues open problems (and sometimes their own!) The problems proposed, together with some of the discussion about them, were recorded and distributed to all participants of the workshop, including the two who were, at the last minute, prevented from attending by circumstances beyond their control.

Felipe Voloch opened the session by asking if most curves are hyperelliptic.

There are several senses in which this question might be interpreted.

Consider the moduli space M_g of genus g curves over $\mathbb{Q}$, and let h be a height function on M_g. Is

$$\lim_{H\to\infty} \frac{\#\{x \in M_g(\mathbb{Q}) : h(x) \le H \text{ and } x \text{ is hyperelliptic}\}}{\#\{x \in M_g(\mathbb{Q}) : h(x) \le H\}} = 1?$$

If so, is there a special locus on M_g whose rational points dominate this count by height? For example, the locus of points corresponding to hyperelliptic curves is of dimension $2g-1$, and the trigonal locus (curves admitting a 3 to 1 morphism to the projective line) is of dimension $2g+1$.

However, this interpretation considers two curves to be isomorphic if they are isomorphic over $\overline{\mathbb{Q}}$. What about $\mathbb{Q}$-isomorphism classes, in which curves are equivalent only if they are isomorphic over $\mathbb{Q}$? There are several ways in which one might count such classes.

First, one might count with respect to naive height in some pre-specified embedding such as a fixed pluricanonical embedding. One might also count by the conductor, or by the Faltings height.

Junjiro Noguchi's question was more technical and specific. Lang's conjecture over number fields states: If X is a projective algebraic variety defined over a number field k and $X_{\mathbf{C}}$ is Kobayashi hyperbolic, then $X(k)$ is finite. An analogue of this over function fields is as follows: Let $f : X \to C$ with a finite subset $S \subset C$ be a morphism of compact varieties with $\dim C = 1$ such that fibers $X_t = f^{-1}t$ are Kobayashi hyperbolic for $t \in C \setminus S$. Moreover, assume a boundary condition (BC) that $X|_{C\setminus S}$ is hyperbolically embedded into X along the boundary fibers $X_t, t \in S$, then the analogue of Lang's conjecture holds (Noguchi, 1985, 1992); i.e., if the set of sections of $f : X \to C$ is Zariski dense, then it is flat, $X \cong C \times X_0$, and there are only finitely many dominant maps from a fixed variety Y onto X_0.

Is it really necessary to assume (BC) or can this hypothesis be omitted? If $\dim X/C = 1$, then (BC) is not necessary.

A second question is the following. Let Y be a fixed compact complex space. Consider $(f; X)$, where X is a compact Kobayashi hyperbolic complex space and $f : Y \to X$ is a surjective holomorphic map. Is the set of such $(f; X)$ finite?

Noam Elkies asked a question more firmly rooted in Diophantine geometry. For which n are there non-trivial maps from $\mathbb{C}$ to the Fermat surface $F_n : w^n + x^n + y^n + z^n = 0$? For $n > 4$ for which the answer is "yes", what are the maps?

For all n, there are trivial maps from $\mathbb{C}$ to F_n, corresponding to lines on the surface. For $n < 4$ or $n = 4$ the surface is rational or K3, and the images of rational maps from $\mathbb{C}$ are dense. For $n > 8$ there are no nontrivial maps, by an argument involving the Wronskian; the same argument excludes nontrivial rational maps for $n = 8$.

For $n = 5$, such maps do exist. Intersect F_5 with the plane $w + x + y + z = 0$, and remove the three lines such as $w + x = y + z = 0$ to get a residual conic. Alas this conic is isomorphic with $\alpha^2 + \beta^2 + \gamma^2 = \mathbb{C}$, and hence has no rational points; still, it is a rational curve, and thus the image of a rational map from $\mathbb{C}$.

For $n = 6$, there are elliptic curves, and thus holomorphic maps, because $(x^2+x-1)^3+(x^2-x-1)^3 = 2(x^6-1)$ and $\mathbb{C}(x, \sqrt{x^2+x-1}, \sqrt{x^2-x-1})$ is a genus-1 function field. For example, there are infinitely many primitive integer solutions of $w^6 + 2x^6 + 125y^6 = 2z^6$, parametrized by an elliptic curve over $\mathbb{Q}$.

Are these maps, and their images under $\mathrm{Aut}(F_n)$, all of the nontrivial maps from $\mathbb{C}$ to F_n for $n = 5, 6$? And what is the answer for $n = 7$ and $n = 8$?

Bjorn Poonen's contribution was the following. Let X be a general genus 2 curve (say, over $\mathbb{C}$). Let $P \in X$ be a general point. Is there an unramified cover $f : Y \to X$ such that there is a nonconstant rational function on Y with divisor supported on $f^{-1}P$?

William Cherry returned the session more firmly to the realm of Nevanlinna theory. Let $X \to B$ be a morphism of smooth varieties such that the generic fiber is a smooth curve of genus $g \ge 2$, and let X_B^n be the n^{th} fiber power of X over B.

A theorem of Caporaso, Harris, and Mazur states that for n large enough, X_B^n dominates a variety of general type. Thus Lang's conjecture implies uniform boundedness of rational points on X: there exists $B(\mathbb{Q}, g)$ such that $\#C(\mathbb{Q}) \leq B(\mathbb{Q}, g)$ for all smooth curves of genus g defined over $\mathbb{Q}$.

Does the Caporaso, Harris, and Mazur correlation theorem have any function theoretic consequences? For example, if X is equipped with a Hermitian metric, does CHM imply some kind of uniform bound on $|f'(0)|_h$ for holomorphic maps f from the unit disc into the fibers X_b of X?

Furthermore, there are theorems in arithmetic geometry that bound the number of solutions to a Diophantine equation without also bounding the height of those solutions. Are there any function-theoretic analogues of any such theorems?

There was much discussion of these questions through the week, and definite progress has been made. In particular, one of the problems was completely solved – by Bjorn Poonen – and we are hopeful that this list of problems will inspire more work in the future.

5 Outcome of the Meeting

The Banff International Research Station is a terrific place to do mathematics. The location is peaceful and inspiring, the staff are efficient, friendly, and helpful, and the facilities are first class. This workshop was one of the first to be organised explicitly around Vojta's conjectures, and was a total success, including the planning of a meeting in 2017 on the same theme by Aaron Levin, David McKinnon, and Thomas Tucker. There has already been success in attacking the problems from the open problem session, and several research collaborations were started or significantly continued during the week. A complete success.

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