Challenges in Infinite Domain Constraint Satisfaction

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Infinite-Domain CSPs

Two Research Directions:

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Part I: ω-categorical templates

The Universal-Algebraic Approach

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2. **Apply the method!**

Permutations

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Finite permutations can be represented by two linear orders:

(3142) corresponds to $(\{a, b, c, d\}; <_1, <_2)$ such that

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- Forb(12)
- Forb(21)
- Forb$(231, 312)$
- Forb$(213, 132)$
- Forb$(\emptyset)$

Forb$(\mathcal{S})$: class of all finite permutations that do not embed any $S \in \mathcal{S}$.

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Challenge 1: Classify CSP(Γ) for all reducts Γ of Δ.

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- The permutations in $\text{Forb}(3142, 2413)$ are called separable in finite combinatorics.

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		x	
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if S_1, S_2 are separable, then

	S_2
S_1	

and

S_1	
	S_2

are separable, too.

An ω-categorical separable template

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Observation (Cameron'02): Expansion by ternary relation $xy|z$ defined by

$$(x <_1 y <_1 z \wedge y <_2 x <_2 z) \vee (z <_1 x <_1 y \wedge z <_2 y <_2 x)$$

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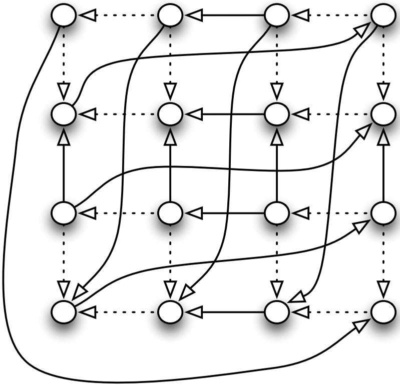
What is the complexity of $\text{CSP}(\Gamma)$?

An Example



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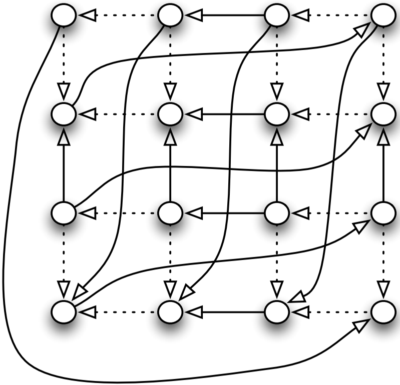

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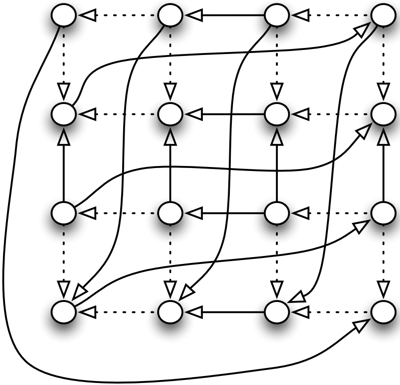

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Hanika's problem CSP(Γ) is not in Datalog

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Atserias+Bulatov+Dawar'07: LFP = Datalog for finite CSPs.
For ω-categorical CSPs: $\text{CSP}(\Gamma) \in \text{LFP} \setminus \text{Datalog}$.

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Challenge 3: For which $\mathcal{S}$ can this problem be formulated as the CSP for an ω-categorical template?

SNP and Fragments

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An SNP τ-sentence is an ESO τ-sentence of the form

$$\exists R_1, \ldots, R_k. \forall x_1, \ldots, x_l. \phi$$

where ϕ is quantifier-free over the signature $\tau \cup \{R_1, \ldots, R_k\}$.

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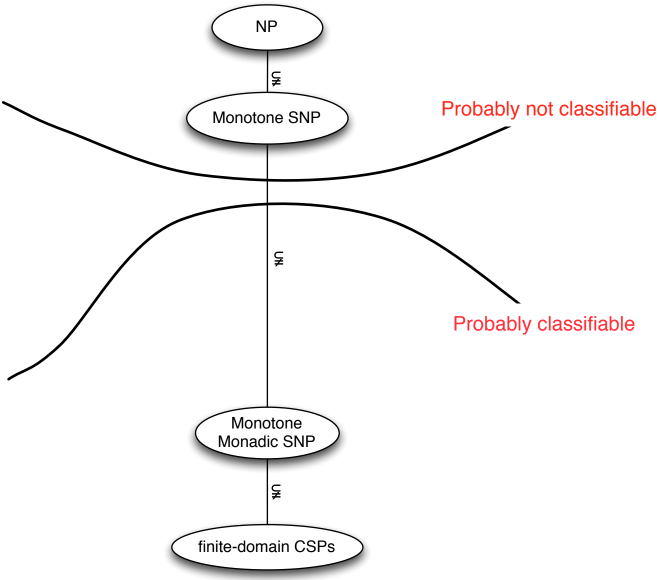
- monadic if $R_1, \dots, R_k$ are unary.

The Results of Feder and Vardi



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Proposition.

An SNP sentence Φ describes a problem of the form CSP(Γ) if and only if Φ is equivalent to a connected monotone SNP sentence.

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Theorem (Feder-Vardi'97,Kun'13).

Every problem in MMSNP is computationally equivalent (under polynomial-time Turing reductions) to a finite domain CSP.

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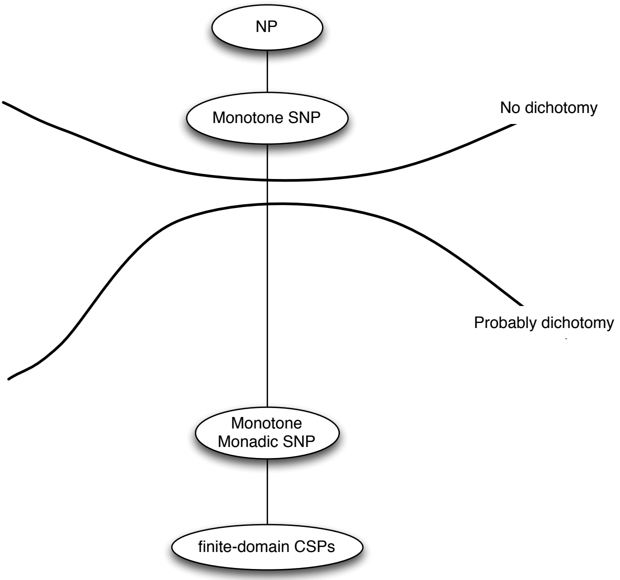

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An MMSNP_2 sentence (over the signature τ) is a monotone SNP sentence such that for every positive literal α there exists a negative input literal containing all variables from α.

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Non-Example:

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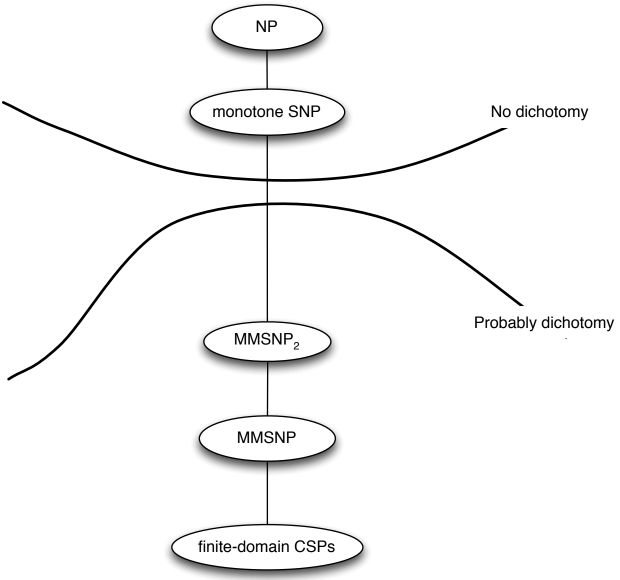
Dichotomy for $MMSNP_2$?



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Theorem (Cherlin+Shelah+Shi'98,MB+Dalmau'06,Madelaine'09).

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Consequence: Canonical function machine can be applied systematically to study MMSNP_2.

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Challenge 5: Give a universal-algebraic characterization for tractability of problems in $MMSNP_1$.

Qualitative vs Quantitative CSPs

Constraint Satisfaction Problems:

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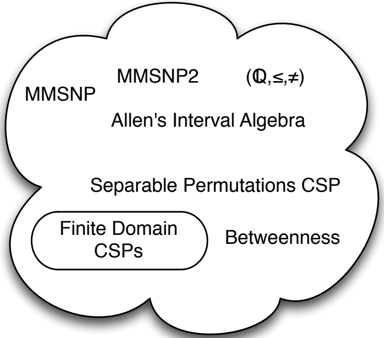

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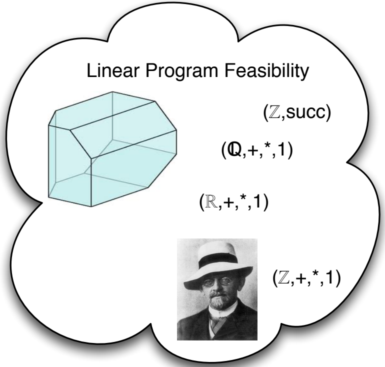

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Challenge 7: Does every semi-algebraic extension of linear program feasibility which is non semilinear simulate a sums-of-roots problem?

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Challenge 9: Do CSPs for reducts of $(\mathbb{Z}; 0, \text{succ})$ have a non-dichotomy?

AAA89 in Dresden

89th Workshop On General Algebra: February 26 (arrival) - March 1, 2015

Invited speakers:

Libor Barto, Georg Grätzer, Maja Pech, Gernot Salzer, John Truss