Classifying Homogeneous Structures

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The first classification theorem

Theorem (Fraïssé's Classification Theorem)

Countable homogeneous structures correspond to amalgamation classes of finite structures.

The Fraïssé limit: $\mathbb{Q} = \lim \mathcal{L}$ ($\mathcal{L}$: finite linear orders).

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Good for existence: is it also good for non-existence? (classification).

"Short answers to simple questions:" Yes.

Longer answer: sometimes . . .

In more detail:

- General theory for the finite case (Lachlan; CSFG meets model theory)
- A few cases of combinatorial interest fully classified, or conjectured
- Some sporadics, and some families, identified via classification
- Cases of particular interest: Ramsey classes

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After a glance at the finite case, I will discuss three cases I have been involved with (2 of them lately):
directed graphs; ordered graphs; graphs as metric spaces

The key: Lachlan's classification of the homogeneous tournaments.

Finite homogeneous graphs

Sheehan 1975, Gardiner 1976

$$C_5,\ E(K_{3,3}),\ m \cdot K_n$$

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Lachlan's view: two sporadics and a set of approximations to $\infty \cdot K_\infty$.

Lachlan's Finiteness Theorem

Given a finite relational language, there are finitely many homogeneous structures Γ_i such that

- The finite homogeneous structures are the homogeneous substructures of the Γ_i.
- The (model-theoretically) stable homogenous structures are the homogeneous substructures.

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Corollary: a stable homogeneous structure can be approximated by finite homogeneous structures. This is not true for the random graph—which can be approximated by finite structures, but not by finite homogeneous ones.

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Division of Labour

Group theory: primitive structures
Model theory: imprimitive structures (modulo primitive)

Binarity Conjecture

What if we bound the relational complexity of the language, but not the number of relations?

> In this metric, cognitive complexity is defined by the *arity* (i.e., number of arguments, or slots) of the relations that are represented by the participant in order to perform the task. An *n*-ary relation is a set of points in
>
> *These findings suggest that a structure defined on four variables is at the limit of human processing capacity.*

Halford et al., *Psychological Science,* (2005)

Binarity Conjecture

Conjecture

A finite primitive homogeneous binary structure is

- *Equality on n points; or*
- *An oriented p-cycle; or*
- *An affine space over a finite field, equipped with an anisotropic quadratic form.*

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Case Division.

Affine	**Non-affine**
(abelian normal subgroup)	(none)
Known	Reduced to almost simple case (Wiscons)

Development

Existence	Henson 1973 ($2^{\aleph_0}$)
Partially ordered sets	Schmerl 1979
Graphs	Lachlan-Woodrow 1980 (induction on amalgamation classes)
Tournaments	Lachlan 1984 (Ramsey argument)
Digraphs	Cherlin 1998 (L/H Smackdown)

Development

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CATALOG

1.	Composite / degenerate	$I_n[T]$, $T[I_n]$
2.	Twisted imprimitive	double covers, generic multipartite, semigeneric
3.	Exceptional primitive	$S(3)$, $\mathcal{P}$, $\mathcal{P}(3)$
4.	Free amalgamation	Omit I_n or tournaments.

The Ramsey Classes

Ramsey precompact expansions of homogeneous directed graphs—*Jakub Jasiński, Claude Laflamme, Lionel Nguyen Van Thé, Robert Woodrow*
(arxiv 24 Oct 2013–23 Jul 2014 (v3))

> *In 2005, Kechris, Pestov and Todorcevic provided a powerful tool . . . More recently, the framework was generalized allowing for further applications, and the purpose of this paper is to apply these new methods in the context of homogeneous directed graphs. In this paper, we show that the age of any homogeneous directed graph allows a Ramsey precompact expansion. Moreover, we . . . describe the respective universal minimal flows* [for $\mathrm{Aut}(\Gamma)$].

LW-L method

(Lachlan/Woodrow 1980, Lachlan 1984)

CATALOG: TOURNAMENTS

Orders	I_1, $\mathbb{Q}$
Local Orders	C_3, $S(2)$
Generic	T_∞

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Case Division

(I) Omit $I\vec{C}_3$: $SL\ldots$hence S or L–1st 4 entries

(I') (Omit $\vec{C}_3 I$: the same.)

(II): Contain $I\vec{C}_3$, $\vec{C}_3 I$—generic (?)

The Main Theorem

Theorem (Lachlan 1984)

Let $\mathcal{A}$ be an amalgamation class of finite tournaments which contains the tournament $I\vec{C}_3$. Then $\mathcal{A}$ contains every finite tournament.

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Definition

Let $\mathcal{A}' = \{A \mid \text{Every } A \cup I \text{ is in } \mathcal{A}\}$

Imagine that we can show that $\mathcal{A}'$ is an amalgamation class containing $I\vec{C}_3$. Then the proof is over!
(By induction on $|A|$.)

Actually, all we need is that $\mathcal{A}'$ *contains* an amalgamation class containing $I\vec{C}_3$.

Induction on Amalgamation classes

Definition

Let $\mathcal{A}^* = \{A \,|\, \text{Every } A \cup L \text{ is in } \mathcal{A}\}$.

This is an amalgamation class contained in $\mathcal{A}$.

Proposition

If $\mathcal{A}$ is an amalgamation class of finite tournaments containing $I\vec{C}_3$, then any tournament of the form $IC_3 \cup L$ is in $\mathcal{A}$.

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Taking stock: The proposition implies the theorem. That is, nearly linear tournaments will give arbitrary tournaments by a soft argument.
(Induction on amalgamation classes.)

Lachlan's Hammer (1984)

Definition

$\mathcal{A}^+ = \{A \,|\, \text{All } L[A] \cup I \text{ are in } \mathcal{A}\}$.

Lemma

$\mathcal{A}^+ \subseteq \mathcal{A}^*$

LACHLAN, with hammer, in 1984 (Artist's conception)

Lachlan's Hammer (1984)

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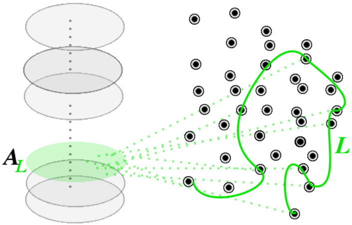

The hammer (Artist's conception)

Mopping up

Lemma

If $\mathcal{A}$ is an amalgamation class of finite tournaments containing $I\vec{C}_3$, then $\mathcal{A}$ contains every 1-point extension of a stack of $\vec{C}_3$'s.

Induction?

The catalog

Nguyen van Thé, 2012: go forth and seek more Ramsey classes among the ordered graphs.

Theorem (2013)

Every homogeneous ordered graph is (and was) known.

The catalog

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Theorem (2013)

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CATALOG

EPO Generic linear extensions of homogeneous partial orders with strong amalgamation.

LT Generic linear orderings of infinite homogeneous tournaments.

LG Generic linear orderings of homogeneous graphs with strong amalgamation.

([EPO] comparability; [LT] “$< = \to$”)

Homogeneous linear extensions of POS

Dolinka and Mašulović 2012: Linear extensions of POS
EPO and homogeneous permutations (Cameron 2002)

Question

Is every primitive homogeneous linearly ordered structure derived from a homogeneous structure without the order?

Open: Triples of linear orders, or beyond.

Conjecture (Minimal form)

A homogeneous primitive k-dimensional permutation is the expansion of a fully generic k'-dimensional permutation by repetition and reversal.

HOG: Case division

$\neg\vec{C}_3^+$	Linear extension of partial order, known
$\neg\vec{C}_3^-$	Complement of the previous
$\vec{C}_3^\pm, \neg(1 \to \vec{C}_3^+)$	Generically ordered local order
$\vec{P}_3^c, (I \perp \vec{P}_3), \vec{I}_\infty$	Generically ordered Henson graph; Lachlan's method [Ch98, Chap. IV].

SURVEYS

Cameron 1998 “A census of infinite distance transitive graphs,” *Discrete Math.* **192** (1998), 11–26.
Diameter δ; bipartite: K_n-free (*cf. KoMePa 1988*)

No doubt, further such variations are possible.

Cherlin 2011 “Two problems on homogeneous structures, revisited,” *in* Contemporary Mathematics **558** (2011).
• Conjectured classification—catalog of variations (cf. KoMePa 1988), supporting evidence.

More Literature

1980 Lachlan/Woodrow: Diameter 2

1980 Cameron: Finite case

1982 Macpherson: locally finite distance transitive case

1988/91 Moss: $\mathbb{U}_N$

1989 Moss: Limit law $\mathrm{Th}(\mathbb{U}_N) \to \mathrm{Th}(\mathbb{U}_{\mathbb{Z}})$

1992 Moss: $\mathbb{U}_\infty$ again, ref. to LW80 and Ca80, asked for a classification

201X AmChMp: Diameter 3 (in preparation)

The Catalog

Komjáth-Mekler-Pach Variations

$$\Gamma^{\delta}_{K,C}$$

Constraints on metric triangles.

- δ: diameter;
- C_0, C_1: bound the perimeter of a triangle of even (resp. odd) length;
- K_- odd perimeter is at least $2K_-$;
- K_+ odd perimeter is at most $2(K_+ + i)$ with i an edge length.

Henson Variations $\Gamma_{\mathcal{S}}$

Forbid $(1, \delta)$-subspaces $\mathcal{S}$ ($\delta \geq 3$).

$$\Gamma^{\delta}_{K,C;\mathcal{S}} \text{ (KMP+H)}$$

Conjecture

Catalog

Exceptional

- Finite
- Tree-like (one part of a $k.\ell$-regular tree with rescaled metric)
- Diameter 2

Generic

- $\Gamma^{\delta}_{K,C;\mathcal{S}}$ for suitable values of δ, K, C
- Antipodal variation $\Gamma^{\delta}_{ap;n}$

Which parameters?

Question

For which choices of parameters δ, K, C (and $\mathcal{S}$) do we actually get an amalgamation class?

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For which choices of parameters δ, K, C (and $\mathcal{S}$) do we actually get an amalgamation class?

Observation. The classes of forbidden triangles are uniformly definable in

$$\mathbb{Z} : +, \leq \text{ (Presburger Arithmetic)}$$

Corollary

For each k, the condition (A_k) that $G^{\delta}_{K,C}$ satisfy amalgamation up to order k is a boolean combination of congruences and inequalities involving $\mathbb{Z}$-linear combinations of the parameters.

A_5

$\delta \geq 2$; $1 \leq K_- \leq K_+ \leq \delta$ or $K_- = \infty$, $K_+ = 0$;
C_0 even, C_1 odd; $2\delta + 1 \leq C_0, C_1 \leq 3\delta + 2$

and

(I) $K_- = \infty$ and $K_+ = 0$, $C_1 = 2\delta + 1$; if $\delta = 2$ then $C' = 8$;

or (II) $K_- < \infty$ and $C \leq 2\delta + K_-$, and

- $\delta \geq 3$;
- $C = 2K_- + 2K_+ + 1$;
- $K_- + K_+ \geq \delta$;
- $K_- + 2K_+ \leq 2\delta - 1$

(IIA) $C' = C + 1$ or

(IIB) $C' > C + 1$, $K_- = K_+$, and $3K_+ = 2\delta - 1$

or (III) $K_- < \infty$ and $C > 2\delta + K_-$, and

- If $\delta = 2$ then $K_+ = 2$;
- $K_- + 2K_+ \geq 2\delta - 1$ and $3K_+ \geq 2\delta$;
- If $K_- + 2K_+ = 2\delta - 1$ then $C \geq 2\delta + K_- + 2$;
- If $C' > C + 1$ then $C \geq 2\delta + K_+$.

Main Lemma

Lemma

If $\mathcal{A}^{\delta}_{C,K}$ has amalgamation up to order 5, then it has amalgamation.

Proof.

Using the conditions on the previous page, define an amalgamation strategy on a case-by-case basis. □

Strategy

r^-, r^+ (lower, upper bounds); $\tilde{r}$ (alternative upper bound)

(I) If $K_- = \infty$: r^-

(II) If $K_- < \infty$ and $C \leq 2\delta + K_-$:

$C' = C + 1$

Case	(a) $r^+ \leq K_+$	(b) $r^- \geq K_-, r^+ > K_+$	(c) $r^- < K_- < K_+ < r^+$
Value	$\min(r^+, \tilde{r})$	r^-	K_+

$C' > C + 1$

Case	(a) $r^+ < K_+$	(b) $r^- > K_+$	(c) $r^- \leq K_+ \leq r^+$
Value	r^+	r^-	$K_+ - \epsilon$ (0 or 1)

$\epsilon = 1$ if $d(a_1, x) = d(a_2, x) = \delta$ (some x)

(III) If $K_- < \infty$ and $C > 2\delta + K_-$:

$C' = C + 1$

Case	(a) $r^- > K_-$	(b) $r^+ \leq K_-$	(c) $r^- \leq K_- < r^+$
Value	r^-	$\min(r^+, \tilde{r})$	$K_- + \epsilon$ (0 or 1)

$\epsilon = 1$ if $K_- + 2K_+ = 2\delta - 1$ and $d(a_1, x) = d(a_2, x) = \delta$ (some x)

$C' > C + 1$

Case	(a) $r^- > K_-$	(b) $r^- \leq K_-, r^+ < K_+$	(c) $r^- \leq K_- < K_+ \leq r^+$
Value	r^-	r^+	$\min(K_+, C - 2\delta - 1)$

Evidence

- Diameter 3
- Full classification of classes determined by constraints of order 3
- Full classification of exceptional types (Γ_1 imprimitive or finite)

Problems

Is the relation $\bigwedge \mathcal{A} \vdash \bigvee \mathcal{B}$ decidable?

- Small languages
  - Homogeneous structures with k linear orders
  - $G + T = T + T$. One family of examples: generic lift of a homogeneous digraph.
  - One asymmetric ternary relation (generalizing tournaments).
- Metric homogeneity
  - Ramsey expansions of Metrically Homogeneous Graphs (Finiteness conjecture for partial metrically homogeneous graphs of bounded diameter)
  - Show that for the classification of metrically homogeneous graphs, finite diameter suffices
- Beyond homogeneity
  - Rado constraints (and Ramsey theory)—decision problems, Ramsey theory.

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