Continuity of homomorphisms to the clone of projections

András Pongrácz

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joint work with Manuel Bodirsky and Michael Pinsker

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The dichotomy conjecture

$\Delta$: a finite relational structure $\text{Pol}(\Delta)$: idempotent clone

Either

$\exists \phi: \text{Pol}(\Delta) \rightarrow 1$ homomorphism,

or

$\exists a$ Taylor (weak n. u., Siggers, cyclic, ...)$)$ operation in $\text{Pol}(\Delta)$.

Conjecture (Bulatov, Jeavons, Krokhin):

item (1) $\Rightarrow$ $\text{CSP}(\Delta)$ is NP-complete (WELL-KNOWN)

item (2) $\Rightarrow$ $\text{CSP}(\Delta)$ $\in P$
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Continuity of homomorphisms

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2. or \(\exists\) a Taylor (weak n. u., Siggers, cyclic, \ldots) operation in \(\text{Pol}(\Delta)\).
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ω-categorical structures

A countable $\Delta$ is ω-categorical if $\forall$ countable $\Gamma$ we have $\text{Th}(\Delta) = \text{Th}(\Gamma) \Rightarrow \Delta \cong \Gamma$.

Examples: finite structures $(\mathbb{Q},<)$, random graph, random tournament, random hypergraphs, generic poset, ... countable vector spaces over $\mathbb{F}_q$, countable atomless Boolean algebra.

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- countable vector spaces over $GF(q)$, countable atomless Boolean algebra
CSP

ω

-categorical templates express more E.g., directed graph acyclicity problem =\(\text{CSP}(Q, <)\).

Question 1: Do we have an analog of the clone dichotomy?

Question 2: What can we say in the two cases about the complexity?
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Topological clones

Pol(∆) is a topological space w.r.t. the pointwise convergence topology.

Ω(a₁↦→b₁,...,aₙ↦→bₙ) := \{ f ∈ Pol(∆) | f(a₁) = b₁,..., f(aₙ) = bₙ \}

B = {Ω(a₁↦→b₁,...,aₙ↦→bₙ)} is a basis.

Bodirsky, Pinsker TFAE for an ω-categorical ∆.

∃Φ : Pol(∆) → 1 continuous homomorphism.

All finite structures have a primitive positive interpretation in ∆ (and in particular, CSP(∆) is NP-hard).
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Pol(Δ) is a topological space w.r.t. the *pointwise convergence* topology.

\[ \Omega(a_1 \mapsto b_1, \ldots, a_n \mapsto b_n) := \{ f \in \text{Pol}(\Delta) \mid f(a_1) = b_1, \ldots, f(a_n) = b_n \} \]

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Bodirsky, Pinsker

TFAE for an \( \omega \)-categorical \( \Delta \).

- \( \exists \Phi : \text{Pol}(\Delta) \to 1 \) \textit{continuous} homomorphism.
- All finite structures have a primitive positive interpretation in \( \Delta \) (and in particular, CSP(\( \Delta \)) is NP-hard).
Main question

Let $\Delta$ be an $\omega$-categorical structure. Assume that $\exists \Phi : \text{Pol}(\Delta) \rightarrow 1$ homomorphism. Does there exist a $\Psi : \text{Pol}(\Delta) \rightarrow 1$ continuous homomorphism?

Theorem. $\exists$ an $\omega$-categorical $\Delta$ such that $\exists \Phi : \text{Pol}(\Delta) \rightarrow 1$ discontinuous homomorphism.

Remark. In that case, there exist continuous ones, too.

Continuity of homomorphisms

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**Theorem.** \( \exists \) an \( \omega \)-categorical \( \Delta \) such that \( \exists \Phi : \text{Pol}(\Delta) \rightarrow \textbf{1} \) discontinuous homomorphism.
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**Theorem.** \( \exists \) an \( \omega \)-categorical \( \Delta \) such that \( \exists \Phi : \text{Pol}(\Delta) \to 1 \) discontinuous homomorphism.

**Remark.** In that case, there exist continuous ones, too.
Proof

\[ \Gamma = (D, R_1, R_2, \ldots) \]

Each \( R_k \) has arity \( k \); it is an equivalence relation on \( k \)-tuples with 2 classes, \((D, R_k)\) is homogeneous, and all the \( R_k \) are freely superimposed. (Cherlin)

\[ \Delta: \text{a reduct of } \Gamma. \]

\[ \Delta = (D, R_1, R_2, \ldots, S_1, S_2, \ldots), \]

where \( S_k \) has arity \( 3^k \), and a triple of \( k \)-tuples is in \( S_k \) iff not all three of them are equivalent w.r.t. \( R_k \).

An \( n \)-ary \( f \in \text{Pol}(\Delta) \) acts on the equivalence classes of \( R_k \) for all \( k \).

This action is an essentially unary function: \( \forall k \in \mathbb{N} \exists 1 \leq i \leq n \) such that it depends on the \( i \)-th coordinate.

\( U \) is a non-principal ultrafilter on \( \mathbb{N} \).

Let \( \Phi(f) = \pi_n^i \) for the unique \( i \) that is the essential coordinate for many \( k \).
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$\Delta$: a reduct of $\Gamma$. $\Delta = (D, R_1, R_2, \ldots, S_1, S_2, \ldots)$, where $S_k$ has arity $3k$, and a triple of $k$-tuples is in $S_k$ iff not all three of them are equivalent w.r.t. $R_k$. 
Proof

Γ = (D, R₁, R₂, ...) 
Each Rₖ has arity 2k; it is an equivalence relation on k-tuples with 2 classes, (D, Rₖ) is homogeneous, and all the Rₖ are freely superimposed. (Cherlin)

Δ: a reduct of Γ. Δ = (D, R₁, R₂, ..., S₁, S₂, ...), where Sₖ has arity 3k, and a triple of k-tuples is in Sₖ iff not all three of them are equivalent w.r.t. Rₖ.

An n-ary f ∈ Pol(Δ) acts on the equivalence classes of Rₖ for all k.
Proof

Γ = (D, R₁, R₂, ⋯)

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Proof

Γ = (D, R_1, R_2, ...)  
Each R_k has arity 2k; it is an equivalence relation on k-tuples with 2 classes, (D, R_k) is homogeneous, and all the R_k are freely superimposed. (Cherlin)  
Δ: a reduct of Γ. Δ = (D, R_1, R_2, ..., S_1, S_2, ...), where S_k has arity 3k, and a triple of k-tuples is in S_k iff not all three of them are equivalent w.r.t. R_k.  

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An n-ary f \in \text{Pol}(Δ) acts on the equivalence classes of R_k for all k. This action is an essentially unary function: \forall k \in \mathbb{N} \exists 1 \leq i \leq n \text{ such that it depends on the } i\text{-th coordinate. } \mathcal{U} \text{ is a non-principal ultrafilter on } \mathbb{N}. \text{ Let } \Phi(f) = \pi^n_i \text{ for the unique } i \text{ that is the essential coordinate for many } k.
Homogeneous structures

A countable relational structure $\Delta$ is homogeneous if $\forall \varphi : A \rightarrow B$ isomorphism between finite substructures $\exists \alpha \in \text{Aut}(\Delta)$ such that $\varphi = \alpha \restriction A$.

If the language of $\Delta$ is finite, then $\Delta$ is $\omega$-categorical.

The tuples $(a_1, \ldots, a_k)$ and $(b_1, \ldots, b_k)$ have the same type in $\Delta$ if they satisfy the same first-order formulas with $k$ free variables.

Fact: $\iff \exists \alpha \in \text{Aut}(\Delta)$ such that $\alpha(a) = b$.

A unary function $f : \Delta \rightarrow \Delta$ is canonical if whenever $a, b \in \Delta$ have the same type, then $f(a)$ and $f(b)$ have the same type (for all $k$).

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A unary function $f : \Delta \to \Delta$ is **canonical** if whenever $\underline{a}, \underline{b} \in \Delta^k$ have the same type, then $f(\underline{a})$ and $f(\underline{b})$ have the same type (for all $k$).
Canonical functions

A unary function $f: \Delta \to \Delta$ is canonical if whenever $a, b \in \Delta^k$ have the same type, then $f(a)$ and $f(b)$ have the same type (for all $k$).

$\iff \forall a \in \Delta^k$ and $\forall \alpha \in \text{Aut}(\Delta) \exists \beta \in \text{Aut}(\Delta)$ such that $f(\alpha(a)) = \beta(f(a))$.

$\iff$ whenever $a_1, b_1 \in \Delta^m$ have the same type, then $f(a_1)$ and $f(b_1)$ have the same type.

An $n$-ary $f: \Delta^n \to \Delta$ is canonical if whenever $a_1, b_1 \in \Delta^k, ..., a_n, b_n \in \Delta^k$ have the same type, then $f(a_1, ..., a_n)$ and $f(b_1, ..., b_n)$ have the same type (for all $k$).

$\iff \forall a_1, ..., a_n \in \Delta^k$ and $\forall \alpha_1, ..., \alpha_n \in \text{Aut}(\Delta) \exists \beta \in \text{Aut}(\Delta)$ such that $f(\alpha_1(a_1), ..., \alpha_n(a_n)) = \beta(f(a_1, ..., a_n))$.

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Δ homogeneous in a finite relational language, maximal arity: \( m \).
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\[ f(\alpha_1(\underline{a}_1), \ldots, \alpha_n(\underline{a}_n)) = \beta(f(\underline{a}_1, \ldots, \underline{a}_n)) \]
\[ \iff \text{whenever } \underline{a}_1, \underline{b}_1 \in \Delta^m, \ldots, \underline{a}_n, \underline{b}_n \in \Delta^m \text{ have the same type, then } f(\underline{a}_1, \ldots, \underline{a}_n) \text{ and } f(\underline{b}_1, \ldots, \underline{b}_n) \text{ have the same type.} \]
The type clone

∆ homogeneous in a finite relational language, maximal arity: $m$. $T_m$: the set of $m$-types.

$C$: a clone of canonical polymorphisms of $∆$ such that $\text{Aut}(∆) \subseteq C$. We say: canonical clone.

$\Phi_{\text{typ}}$ maps an $n$-ary $f \in C$ to the corresponding $n$-ary function on $m$-types. The image of $\Phi_{\text{typ}}$ is the type clone of $C$.

Claim. $\Phi_{\text{typ}}$ is a continuous homomorphism.

Proof. The $\Phi_{\text{typ}}$-image of an $n$-ary $f$ depends only on the restriction of $f$ to a big enough finite set.
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\( \Delta \) homogeneous in a finite relational language, maximal arity: \( m \).

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**Claim.** \( \Phi^{\text{typ}} \) is a continuous homomorphism.
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Claim. \( \Phi^{\text{typ}} \) is a continuous homomorphism.

Proof. The \( \Phi^{\text{typ}} \)-image of an \( n \)-ary \( f \) depends only on the restriction of \( f \) to a big enough finite set.
Theorem. Let $\Delta$ be a homogeneous structure in a finite relational language, and let $C$ be a closed canonical clone. Assume that there exists an $n$-ary cyclic operation $f$ in the type clone, i.e., $f(x_1,\ldots,x_n) = f(x_2,\ldots,x_n,x_1) = \cdots = f(x_n,x_1,\ldots,x_{n-1})$.

Then there exists an $n$-ary $g \in C$ and unary functions $\alpha_1,\ldots,\alpha_n \in C$ such that $\alpha_1 \circ g(x_1,\ldots,x_n) = \alpha_2 \circ g(x_2,\ldots,x_n,x_1) = \cdots = \alpha_n \circ g(x_n,x_1,\ldots,x_{n-1})$.
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Theorem. Let $\Delta$ be a homogeneous structure in a finite relational language, and let $\mathcal{C}$ be a closed canonical clone. If $\exists \Phi : \mathcal{C} \to 1$ homomorphism, then there is also a continuous $\mathcal{C} \to 1$ homomorphism.
**Theorem.** Let $\Delta$ be a homogeneous structure in a finite relational language, and let $C$ be a closed canonical clone. Assume that there exists an $n$-ary cyclic operation $f$ in the type clone, i.e.,

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**Theorem.** Let $\Delta$ be a homogeneous structure in a finite relational language, and let $\mathcal{C}$ be a closed canonical clone. If $\exists \Phi : \mathcal{C} \rightarrow 1$ homomorphism, then there is also a continuous $\mathcal{C} \rightarrow 1$ homomorphism.
Proof. If the type clone does not map homomorphically to \( \mathbb{1} \), then it contains a cyclic operation \( f \). This \( f \) lifts to an operation \( g \in C \) that is cyclic modulo unaries, a contradiction. Thus the type clone maps to \( \mathbb{1} \) via a (continuous) homomorphism \( \zeta \). Then \( \zeta \circ \Phi : C \to \mathbb{1} \) is a continuous homomorphism.

Corollary 1. The clone dichotomy holds for closed canonical clones.

Corollary 2. Let \( \Delta \) be homogeneous in a finite relational language. Assume that \( C = \text{Pol}(\Delta) \) is a closed canonical clone, and that \( \exists \Phi : C \to \mathbb{1} \) homomorphism. Then \( \text{CSP}(\Delta) \) is NP-hard.
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**Corollary 1.** The clone dichotomy holds for closed canonical clones.

**Corollary 2.** Let $\Delta$ be homogeneous in a finite relational language. Assume that $\mathcal{C} = \text{Pol}(\Delta)$ is a closed canonical clone, and that $\exists \Phi : \mathcal{C} \to 1$ homomorphism. Then CSP$(\Delta)$ is NP-hard.
If $\Delta$ has a first-order definition in a homogeneous relational structure $\Gamma$ that is ordered and has the Ramsey property, then every function generates a canonical one. Moreover, $\text{Pol}(\Delta) = \bigcup_{i=1}^{\infty} C_i$, where each $C_i$ is a closed canonical clone.
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Moreover, $\text{Pol}(\Delta) = \bigcup_{i=1}^{\infty} C_i$, where each $C_i$ is a closed canonical clone.
Let $\Delta$ be first-order definable in $\langle \mathbb{Q}, < \rangle$. Then $\text{CSP}(\Delta)$ is either in $\text{P}$ or $\text{NP}$ – complete.
Bodirsky, Kára

Let $\Delta$ be first-order definable in $(\mathbb{Q},<)$. Then CSP($\Delta$) is either in P or NP – complete.

Bodirsky, Pinsker

Let $\Delta$ be first-order definable in the random graph. Then CSP($\Delta$) is either in P or NP – complete.
Some open problems

- Is there a closed clone $\mathcal{C}$ with a homomorphism to $1$ but no continuous one?
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- Is there a model of ZF in which every homomorphism from a closed clone to $\mathbf{1}$ is continuous?