Optimal strong Maltsev conditions for congruence meet-semidistributivity

Matthew Moore

Vanderbilt University

November 27, 2014

Joint work with...

Jelena Jovanović, Belgrade University

Petar Marković, University of Novi Sad

Ralph McKenzie, Vanderbilt University

SD(∧) varieties

Definition

A variety $\mathcal{V}$ is **congruence meet-semidistributive** (SD(∧)) if for every algebra $\mathbb{A} \in \mathcal{V}$,

$$\mathrm{Con}(\mathbb{A}) \models \big[(x \wedge y \approx x \wedge z) \to (x \wedge y \approx x \wedge (y \vee z)) \big].$$

SD(∧) varieties

Definition

A variety $\mathcal{V}$ is **congruence meet-semidistributive** (SD(∧)) if for every algebra $\mathbb{A} \in \mathcal{V}$,

$$\mathrm{Con}(\mathbb{A}) \models \left[(x \wedge y \approx x \wedge z) \to (x \wedge y \approx x \wedge (y \vee z)) \right].$$

(for $\mathcal{V}$ locally finite ...)

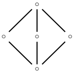
SD($\wedge$) varieties

Definition

A variety $\mathcal{V}$ is **congruence meet-semidistributive** (SD($\wedge$)) if for every algebra $\mathbb{A} \in \mathcal{V}$,

$$\mathrm{Con}(\mathbb{A}) \models \big[(x \wedge y \approx x \wedge z) \rightarrow (x \wedge y \approx x \wedge (y \vee z)) \big].$$

(for $\mathcal{V}$ locally finite ...)

- ($\Leftrightarrow$)  is not a sublattice of $\mathrm{Con}(\mathbb{A})$ for all $\mathbb{A} \in \mathcal{V}$

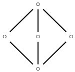
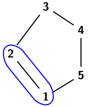
SD(∧) varieties

Definition

A variety $\mathcal{V}$ is **congruence meet-semidistributive** (SD(∧)) if for every algebra $\mathbb{A} \in \mathcal{V}$,

$$\mathrm{Con}(\mathbb{A}) \models \big[(x \wedge y \approx x \wedge z) \rightarrow (x \wedge y \approx x \wedge (y \vee z)) \big].$$

(for $\mathcal{V}$ locally finite ...)

- (⇔)  is not a sublattice of $\mathrm{Con}(\mathbb{A})$ for all $\mathbb{A} \in \mathcal{V}$
- (⇔) $\mathcal{V}$ omits TCT types **1** and **2**: 

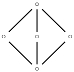
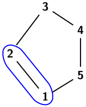
SD(∧) varieties

Definition

A variety $\mathcal{V}$ is **congruence meet-semidistributive** (SD(∧)) if for every algebra $\mathbb{A} \in \mathcal{V}$,

$$\mathrm{Con}(\mathbb{A}) \models \big[(x \wedge y \approx x \wedge z) \to (x \wedge y \approx x \wedge (y \vee z)) \big].$$

(for $\mathcal{V}$ locally finite ...)

- (⇔)  is not a sublattice of $\mathrm{Con}(\mathbb{A})$ for all $\mathbb{A} \in \mathcal{V}$
- (⇔) $\mathcal{V}$ omits TCT types **1** and **2**: 

- (⇔) $\mathrm{Con}(\mathbb{A}) \models [x, y] \approx x \wedge y$ for any $\mathbb{A} \in \mathcal{V}$ (*congruence neutral*)

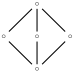
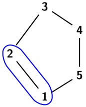
SD(∧) varieties

Definition

A variety $\mathcal{V}$ is **congruence meet-semidistributive** (SD(∧)) if for every algebra $\mathbb{A} \in \mathcal{V}$,

$$\mathrm{Con}(\mathbb{A}) \models \big[(x \wedge y \approx x \wedge z) \to (x \wedge y \approx x \wedge (y \vee z)) \big].$$

(for $\mathcal{V}$ locally finite ...)

- (⇔)  is not a sublattice of $\mathrm{Con}(\mathbb{A})$ for all $\mathbb{A} \in \mathcal{V}$
- (⇔) $\mathcal{V}$ omits TCT types **1** and **2**: 

- (⇔) $\mathrm{Con}(\mathbb{A}) \models [x, y] \approx x \wedge y$ for any $\mathbb{A} \in \mathcal{V}$ (*congruence neutral*)
- Park's Conjecture is true (if $\mathcal{V}$ has finite residual bound, then $\mathcal{V}$ is finitely based) [Willard 2000]

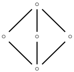
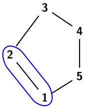
SD(∧) varieties

Definition

A variety $\mathcal{V}$ is **congruence meet-semidistributive** (SD(∧)) if for every algebra $\mathbb{A} \in \mathcal{V}$,

$$\mathrm{Con}(\mathbb{A}) \models \big[(x \wedge y \approx x \wedge z) \to (x \wedge y \approx x \wedge (y \vee z)) \big].$$

(for $\mathcal{V}$ locally finite ...)

- (⇔)  is not a sublattice of $\mathrm{Con}(\mathbb{A})$ for all $\mathbb{A} \in \mathcal{V}$
- (⇔) $\mathcal{V}$ omits TCT types **1** and **2**: 

- (⇔) $\mathrm{Con}(\mathbb{A}) \models [x, y] \approx x \wedge y$ for any $\mathbb{A} \in \mathcal{V}$ (*congruence neutral*)
- Park's Conjecture is true (if $\mathcal{V}$ has finite residual bound, then $\mathcal{V}$ is finitely based) [Willard 2000]
- CSP($\mathbb{A}$) can be solved using local consistency checking [Barto 2014]

CSP($\mathbb{A}$)

Definition

For a finite algebra $\mathbb{A}$, CSP($\mathbb{A}$) is the CSP restricted to constraints C such that $C \leq \mathbb{A}^n$ for some n.

CSP($\mathbb{A}$)

Definition

For a finite algebra $\mathbb{A}$, CSP($\mathbb{A}$) is the CSP restricted to constraints C such that $C \leq \mathbb{A}^n$ for some n.

Examine relations over $\mathbb{F}^{\mathcal{V}}(x_1, \ldots, x_n)$.

CSP($\mathbb{A}$)

Theorem (Barto 2014)

If $\mathbb{A}$ is idempotent and $\mathcal{V}(\mathbb{A})$ is SD($\wedge$), then every $(2,3)$-minimal instance of CSP($\mathbb{A}$) has a solution.

CSP($\mathbb{A}$)

Theorem (Barto 2014)

If $\mathbb{A}$ is idempotent and $\mathcal{V}(\mathbb{A})$ is SD($\wedge$), then every $(2, 3)$-minimal instance of CSP($\mathbb{A}$) has a solution.

Definition

Let $(V; A; \mathcal{C})$ be a CSP instance.

- $(V; A; \mathcal{C})$ is **2-consistent** if for every $U \subseteq V$ with $|U| \leq 2$ and every pair of constraints $C, D \in \mathcal{C}$ containing U in their scopes, $C|_U = D|_U$.

CSP($\mathbb{A}$)

Theorem (Barto 2014)

If $\mathbb{A}$ is idempotent and $\mathcal{V}(\mathbb{A})$ is SD($\wedge$), then every $(2,3)$-minimal instance of CSP($\mathbb{A}$) has a solution.

Definition

Let $(V; A; \mathcal{C})$ be a CSP instance.

- $(V; A; \mathcal{C})$ is **2-consistent** if for every $U \subseteq V$ with $|U| \leq 2$ and every pair of constraints $C, D \in \mathcal{C}$ containing U in their scopes, $C|_U = D|_U$.
- $(V; A; \mathcal{C})$ is $(\mathbf{2}, \mathbf{3})$-**minimal** if it is 2-consistent and every subset $U \leq V$ with $|U| \leq 3$ is contained in the scope of some constraint.

Theorem

$\mathcal{V}$ is SD($\wedge$) iff $\mathcal{V}$ satisfies an idempotent Maltsev conditions which fails in any variety of modules.

Some known Maltsev characterizations

A variety $\mathcal{V}$ is said to satisfy WNU(n) if it has an idempotent n-ary term $t(\cdots)$ such that

$$\mathcal{V} \models t(y, x, \ldots, x) \approx t(x, y, x, \ldots, x) \approx \cdots \approx t(x, \ldots, x, y).$$

This is the weak near unanimity term condition.

Some known Maltsev characterizations

A variety $\mathcal{V}$ is said to satisfy WNU(n) if it has an idempotent n-ary term $t(\cdots)$ such that

$$\mathcal{V} \models t(y, x, \ldots, x) \approx t(x, y, x, \ldots, x) \approx \cdots \approx t(x, \ldots, x, y).$$

This is the weak near unanimity term condition.

TFAE for locally finite $\mathcal{V}$

- $\mathcal{V}$ is SD($\wedge$)

Some known Maltsev characterizations

A variety $\mathcal{V}$ is said to satisfy WNU(n) if it has an idempotent n-ary term $t(\cdots)$ such that

$$\mathcal{V} \models t(y, x, \ldots, x) \approx t(x, y, x, \ldots, x) \approx \cdots \approx t(x, \ldots, x, y).$$

This is the weak near unanimity term condition.

TFAE for locally finite $\mathcal{V}$

- $\mathcal{V}$ is SD($\wedge$)
- there exists $n > 1$ such that $\mathcal{V} \models$ WNU(k) for all $k \geq n$ [Maroti, McKenzie 2008]

Some known Maltsev characterizations

A variety $\mathcal{V}$ is said to satisfy WNU(n) if it has an idempotent n-ary term $t(\cdots)$ such that

$$\mathcal{V} \models t(y, x, \ldots, x) \approx t(x, y, x, \ldots, x) \approx \cdots \approx t(x, \ldots, x, y).$$

This is the weak near unanimity term condition.

TFAE for locally finite $\mathcal{V}$

- $\mathcal{V}$ is SD($\wedge$)
- there exists $n > 1$ such that $\mathcal{V} \models$ WNU(k) for all $k \geq n$ [Maroti, McKenzie 2008]
- $\mathcal{V}$ satisfies WNU(4) via $t(\cdots)$ and WNU(3) via $s(\cdots)$ and

$$t(y, x, x, x) \approx s(y, x, x)$$

  [Kozik, Krokhin, Valeriote, Willard 2013]

“Better” Maltsev conditions

Let Σ and Ω be strong Maltsev conditions.

“Better” Maltsev conditions

Let Σ and Ω be strong Maltsev conditions.
(finite sets of equations in some language)

- Write $\Sigma \preceq \Omega$ if any variety which realizes Ω must also realize Σ.

“Better” Maltsev conditions

Let Σ and Ω be strong Maltsev conditions.
(finite sets of equations in some language)

- Write $\Sigma \preceq \Omega$ if any variety which realizes Ω must also realize Σ.
- This induces a preorder.
- If $\Sigma \preceq \Omega$, we say Ω is **stronger** than Σ.

"Better" Maltsev conditions

Let Σ and Ω be strong Maltsev conditions.
(finite sets of equations in some language)

- Write $\Sigma \preceq \Omega$ if any variety which realizes Ω must also realize Σ.
- This induces a preorder.
- If $\Sigma \preceq \Omega$, we say Ω is **stronger** than Σ.
- If $\Sigma \preceq \Omega \preceq \Sigma$, we say the conditions are **equivalent** and write $\Sigma \sim \Omega$.

"Better" Maltsev conditions

Let Σ and Ω be strong Maltsev conditions.
(finite sets of equations in some language)

- Write $\Sigma \preceq \Omega$ if any variety which realizes Ω must also realize Σ.
- This induces a preorder.
- If $\Sigma \preceq \Omega$, we say Ω is **stronger** than Σ.
- If $\Sigma \preceq \Omega \preceq \Sigma$, we say the conditions are **equivalent** and write $\Sigma \sim \Omega$.

Many strong Maltsev conditions which are not equivalent are equivalent **within the class** of locally finite varieties.

Characterizations of SD(∧) for locally finite $\mathcal{V}$

$t(\dots), s(\dots)$ WNU's $t(yxxx) \approx s(yxx)$	$\exists n \forall k > n$ there is k-ary WNU

A restricted $\preceq$-minimal characterization

Theorem (JMMM)

A locally finite variety $\mathcal{V}$ is SD($\wedge$) iff there are idempotent terms $p(\cdots)$, $q(\cdots)$ such that

$$p(x,x,y) \approx p(x,y,x) \approx p(y,x,x) \approx q(x,y,z) \text{ and}$$
$$q(x,x,y) \approx q(x,y,y)$$

A restricted $\preceq$-minimal characterization

Theorem (JMMM)

A locally finite variety $\mathcal{V}$ is SD($\wedge$) iff there are idempotent terms $p(\cdots)$, $q(\cdots)$ such that

$$p(x,x,y) \approx p(x,y,x) \approx p(y,x,x) \approx q(x,y,z) \text{ and}$$
$$q(x,x,y) \approx q(x,y,y)$$

There is no idempotent strong Maltsev condition characterizing SD($\wedge$) in the language with one ternary and any number of binary operation symbols.

A restricted $\preceq$-minimal characterization

Theorem (JMMM)

A locally finite variety $\mathcal{V}$ is SD($\wedge$) iff there are idempotent terms $p(\cdots)$, $q(\cdots)$ such that

$$p(x,x,y) \approx p(x,y,x) \approx p(y,x,x) \approx q(x,y,z) \text{ and}$$
$$q(x,x,y) \approx q(x,y,y)$$

There is no idempotent strong Maltsev condition characterizing SD($\wedge$) in the language with one ternary and any number of binary operation symbols.

In the class of all strong idempotent Maltsev conditions in a language consisting of 2 ternary operation symbols, a computer search produced as a candidate for being $\preceq$-minimal for characterizing SD($\wedge$) varieties. [Jovanović 2013]

Characterizations of SD($\wedge$) for locally finite $\mathcal{V}$

Language =
two 3-ary ops

$p(xxy) \approx p(xyx)$
$\approx p(yxx) \approx q(xyz)$
$\approx q(xxy) \approx q(xyy)$

$t(\dots), s(\dots)$ WNU's
$t(yxxx) \approx s(yxx)$

$\exists n \forall k > n$ there
is k-ary WNU

~~Language =
one 3-ary, any # binary~~

Other optimal Maltsev characterizations

Theorem (JMMM)

A locally finite variety $\mathcal{V}$ is SD($\wedge$) iff there is an idempotent term $t(\cdots)$ such that

$$
\begin{aligned}
t(y,x,x,x) &\approx t(x,y,x,x) \approx t(x,x,y,x) \approx t(x,x,x,y) \\
&\approx t(y,y,x,x) \approx t(y,x,y,x) \approx t(x,y,y,x)
\end{aligned}
$$

Other optimal Maltsev characterizations

Theorem (JMMM)

A locally finite variety $\mathcal{V}$ is SD($\wedge$) iff there is an idempotent term $t(\cdots)$ such that

$$\begin{aligned} t(y,x,x,x) &\approx t(x,y,x,x) \approx t(x,x,y,x) \approx t(x,x,x,y) \\ &\approx t(y,y,x,x) \approx t(y,x,y,x) \approx t(x,y,y,x) \end{aligned}$$

Look at the relation

$$U = \mathrm{Sg}\begin{pmatrix} x & x & x & y \\ x & x & y & x \\ x & y & x & x \\ y & x & x & x \\ y & y & x & x \\ y & x & y & x \\ x & y & y & x \end{pmatrix}$$

in $\mathbb{F}^{\mathcal{V}}(x,y)$,

Other optimal Maltsev characterizations

Theorem (JMMM)

A locally finite variety $\mathcal{V}$ is SD($\wedge$) iff there is an idempotent term $t(\cdots)$ such that

$$\begin{aligned} t(y,x,x,x) &\approx t(x,y,x,x) \approx t(x,x,y,x) \approx t(x,x,x,y) \\ &\approx t(y,y,x,x) \approx t(y,x,y,x) \approx t(x,y,y,x) \end{aligned}$$

Look at the relation

$$U = \mathrm{Sg}\begin{pmatrix} x & x & x & y \\ x & x & y & x \\ x & y & x & x \\ y & x & x & x \\ y & y & x & x \\ y & x & y & x \\ x & y & y & x \end{pmatrix}$$

in $\mathbb{F}^{\mathcal{V}}(x,y)$, plus 11 ternary relations,

Other optimal Maltsev characterizations

Theorem (JMMM)

A locally finite variety $\mathcal{V}$ is SD($\wedge$) iff there is an idempotent term $t(\cdots)$ such that

$$\begin{aligned} t(y,x,x,x) &\approx t(x,y,x,x) \approx t(x,x,y,x) \approx t(x,x,x,y) \\ &\approx t(y,y,x,x) \approx t(y,x,y,x) \approx t(x,y,y,x) \end{aligned}$$

Look at the relation

$$U = \mathrm{Sg}\begin{pmatrix} x & x & x & y \\ x & x & y & x \\ x & y & x & x \\ y & x & x & x \\ y & y & x & x \\ y & x & y & x \\ x & y & y & x \end{pmatrix}$$

in $\mathbb{F}^{\mathcal{V}}(x,y)$, plus 11 ternary relations, plus 3 binary.

Other optimal Maltsev characterizations

Theorem (JMMM)

A locally finite variety $\mathcal{V}$ is SD($\wedge$) iff there is an idempotent term $t(\cdots)$ such that

$$t(y,x,x,x) \approx t(x,y,x,x) \approx t(x,x,y,x) \approx t(x,x,x,y) \approx t(y,y,x,x) \approx t(y,x,y,x) \approx t(x,y,y,x)$$

Look at the relation

$$U = \mathrm{Sg}\begin{pmatrix} x & x & x & y \\ x & x & y & x \\ x & y & x & x \\ y & x & x & x \\ y & y & x & x \\ y & x & y & x \\ x & y & y & x \end{pmatrix}$$

in $\mathbb{F}^{\mathcal{V}}(x,y)$, plus 11 ternary relations, plus 3 binary. Then use a (difficult) Ramsey argument. **Can we do better?**

How much better can we do?

Theorem

Any strong Maltsev condition of the form

$$f(x, \ldots, x) \approx x \qquad \text{and} \qquad f(y_1, \ldots, y_n) \approx f(z_1, \ldots, z_n),$$

where $y_i, z_j \in \{x_1, \ldots, x_m\}$*, that is realized in a nontrivial semilattice can also be realized in a nontrivial module.*

Characterizations of SD(∧) for locally finite $\mathcal{V}$

Language = two 3-ary ops

$t(yxxx) \approx t(xyxx)$
$\approx t(xxyx) \approx t(xxxy)$
$\approx t(yyxx) \approx t(xyyx)$

$p(xxy) \approx p(xyx)$
$\approx p(yxx) \approx q(xyz)$
$\approx q(xxy) \approx q(xyy)$

$t(\ldots), s(\ldots)$ WNU's
$t(yxxx) \approx s(yxx)$

$\exists n \forall k > n$ there is k-ary WNU

Language = one 3-ary, any # binary

Language = idempotent f, $f(\overline{x}) = f(\overline{y})$

Candidates for "least-equations"-optimal

Amongst all idempotent strong Maltsev conditions of the form

$$f(\overline{x}) \approx f(\overline{y}) \approx f(\overline{z}),$$

for $f(\cdots)$ of arity ≤ 4, a computer search eliminates all but two candidates:

Candidates for "least-equations"-optimal

Amongst all idempotent strong Maltsev conditions of the form

$$f(\overline{x}) \approx f(\overline{y}) \approx f(\overline{z}),$$

for $f(\cdots)$ of arity ≤ 4, a computer search eliminates all but two candidates:

$$t\begin{pmatrix} x & x & y & z \\ y & z & y & x \\ x & z & z & y \end{pmatrix} = \begin{pmatrix} w \\ w \\ w \end{pmatrix}$$

Candidates for “least-equations”-optimal

Amongst all idempotent strong Maltsev conditions of the form

$$f(\overline{x}) \approx f(\overline{y}) \approx f(\overline{z}),$$

for $f(\cdots)$ of arity ≤ 4, a computer search eliminates all but two candidates:

$$t\begin{pmatrix} x & x & y & z \\ y & z & y & x \\ x & z & z & y \end{pmatrix} = \begin{pmatrix} w \\ w \\ w \end{pmatrix} \qquad t\begin{pmatrix} x & x & y & z \\ y & x & z & x \\ y & z & x & y \end{pmatrix} = \begin{pmatrix} w \\ w \\ w \end{pmatrix}$$

Candidates for "least-equations"-optimal

Amongst all idempotent strong Maltsev conditions of the form

$$f(\overline{x}) \approx f(\overline{y}) \approx f(\overline{z}),$$

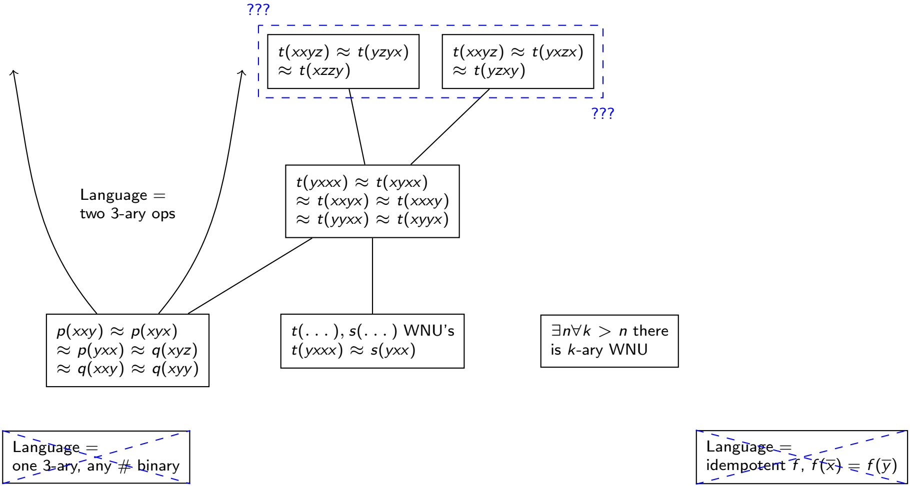
for $f(\cdots)$ of arity ≤ 4, a computer search eliminates all but two candidates:

$$t\begin{pmatrix} x & x & y & z \\ y & z & y & x \\ x & z & z & y \end{pmatrix} = \begin{pmatrix} w \\ w \\ w \end{pmatrix} \qquad t\begin{pmatrix} x & x & y & z \\ y & x & z & x \\ y & z & x & y \end{pmatrix} = \begin{pmatrix} w \\ w \\ w \end{pmatrix}$$

Problem

Prove that a locally finite SD(∧) variety satisfies one (or both) of the Maltsev conditions above.

Characterizations of SD(∧) for locally finite $\mathcal{V}$



WNU's (special and otherwise)

Theorem (JMMM)

A locally finite variety $\mathcal{V}$ is SD($\wedge$) iff there is a term $t(x, y)$ and for all $n \geq 3$,

- *there exists n-ary WNU, $w(\cdots)$ and*
- *$t(x, y) = w(y, x, \dots, x)$.*

WNU's (special and otherwise)

Theorem (JMMM)

A locally finite variety $\mathcal{V}$ is SD($\wedge$) iff there is a term $t(x,y)$ and for all $n \geq 3$,

- *there exists n-ary WNU, $w(\cdots)$ and*
- $t(x,y) = w(y,x,\ldots,x)$.

A WNU $w(\cdots)$ is called **special** if $t(x,t(x,y)) = t(x,y)$ for $t(x,y) = w(y,x,\ldots,x)$.

WNU's (special and otherwise)

Theorem (JMMM)

A locally finite variety $\mathcal{V}$ is SD($\wedge$) iff there is a term $t(x,y)$ and for all $n \geq 3$,

- *there exists n-ary WNU, $w(\cdots)$ and*
- $t(x,y) = w(y,x,\ldots,x)$.

A WNU $w(\cdots)$ is called **special** if $t(x,t(x,y)) = t(x,y)$ for $t(x,y) = w(y,x,\ldots,x)$.

Problem

Prove that the WNU's in the above theorem can be taken to be special.

WNU's (special and otherwise)

Theorem (JMMM)

A locally finite variety $\mathcal{V}$ is SD($\wedge$) iff there is a term $t(x,y)$ and for all $n \geq 3$,

- *there exists n-ary WNU, $w(\cdots)$ and*
- $t(x,y) = w(y,x,\ldots,x)$.

A WNU $w(\cdots)$ is called **special** if $t(x,t(x,y)) = t(x,y)$ for $t(x,y) = w(y,x,\ldots,x)$.

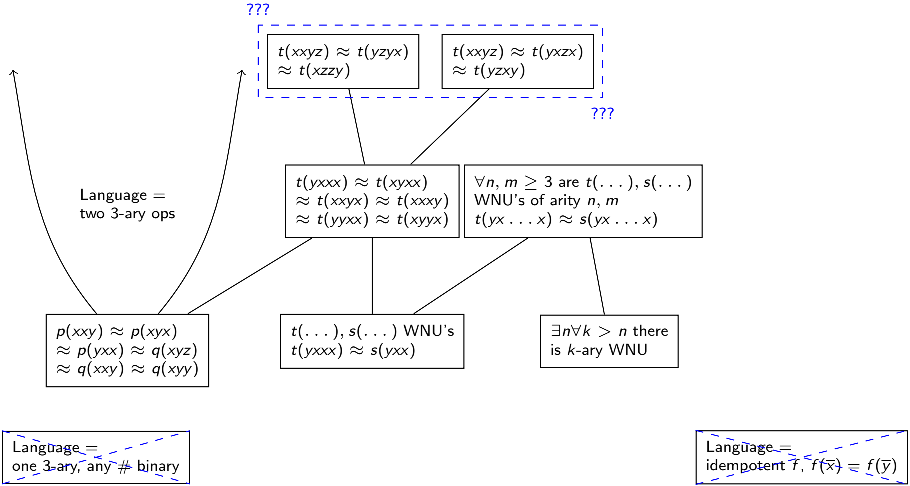
Problem

Prove that the WNU's in the above theorem can be taken to be special.

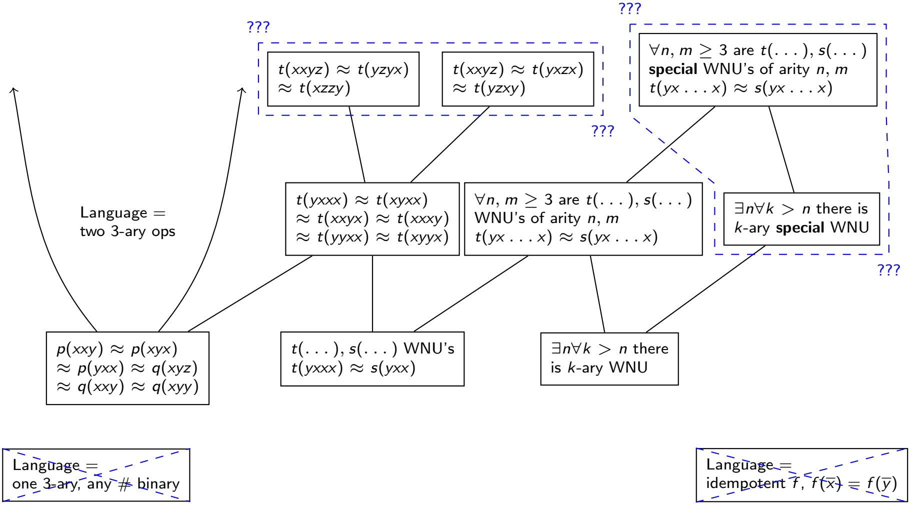
Problem

A locally finite variety $\mathcal{V}$ is SD($\wedge$) if there exists n such that $\mathcal{V}$ has special WNU's of all arities $k > n$.

Characterizations of SD(∧) for locally finite $\mathcal{V}$



Characterizations of SD(∧) for locally finite $\mathcal{V}$



Thank you.

Thank you.

Shanks workshop: Open Problems in Universal Algebra

Vanderbilt University

May 28 – June 1, 2015

www.math.vanderbilt.edu/~moorm10/shanks/