Constraint Satisfaction on Infinite Domains

3rd and last session

Michael Pinsker

Technische Universität Wien / Université Diderot - Paris 7

Funded by FWF grant I836-N23

Algebraic and Model Theoretical Methods in Constraint Satisfaction

Banff International Research Station

2014

Outline reminder

Part I: CSPs / dividing the world /
pp definitions, polymorphism clones, ω-categoricity

Part II: pp interpretations / topological clones

Part III: Canonical functions, Ramsey structures / Graph-SAT

Part IV: Model-complete cores / The infinite tractability conjecture

Canonizing functions on Ramsey structures

Canonizing functions on Ramsey structures

Proposition (Bodirsky + MP + Tsankov '11)

Let

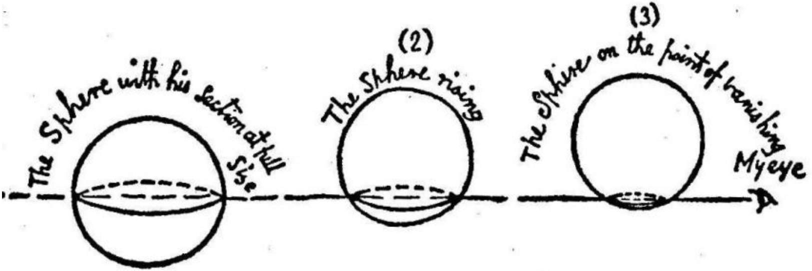
- Δ be ordered Ramsey homogeneous finite relational language
- $f : \Delta^n \to \Delta$
- $c_1, \ldots, c_k \in \Delta$.

Then

$$\overline{\{\beta(f(\alpha_1(x_1), \ldots, \alpha_n(x_n))) \mid \beta, \alpha_i \in \mathrm{Aut}(\Delta)\}}$$

contains a function which

- is canonical as a function on $(\Delta, c_1, \ldots, c_k)$
- is identical with f on $\{c_1, \ldots, c_k\}^n$.


The Sphere with his section at full Size
(2)
The Sphere rising
(3)
The Sphere on the point of vanishing
My eye

Graph-SAT

Complexity of CSP for reducts of G

Complexity of CSP for reducts of G

Theorem (Bodirsky + MP '10)

Let Γ be a reduct of the random graph. Then:

- Either Γ has one out of 17 canonical polymorphisms, and CSP(Γ) is tractable,
- or CSP(Γ) is NP-complete.

Complexity of CSP for reducts of G

Theorem (Bodirsky + MP '10)

Let Γ be a reduct of the random graph. Then:

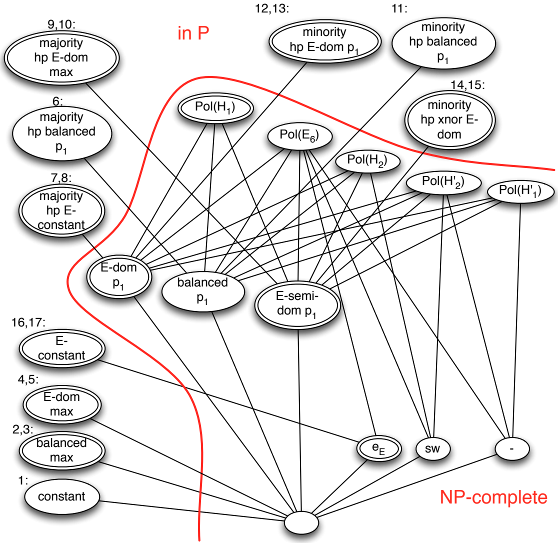
- Either Γ has one out of 17 canonical polymorphisms, and CSP(Γ) is tractable,
- or CSP(Γ) is NP-complete.

Theorem (Bodirsky + MP '10)

Let Γ be a reduct of the random graph. Then:

- Either Γ pp-defines one out of 5 hard relations, and CSP(Γ) is NP-complete,
- or CSP(Γ) is tractable.

Graph-SAT classification



Theorem

The following 17 distinct clones are precisely the minimal tractable closed function clones containing Aut(G):

1. The clone generated by a constant operation.
2. The clone generated by a balanced binary injection of type max.
3. The clone generated by a balanced binary injection of type min.
4. The clone generated by an E-dominated binary injection of type max.
5. The clone generated by an N-dominated binary injection of type min.
6. The clone generated by a function of type majority which is hyperplanely balanced and of type projection.
7. The clone generated by a function of type majority which is hyperplanely E-constant.
8. The clone generated by a function of type majority which is hyperplanely N-constant.
9. The clone generated by a function of type majority which is hyperplanely of type max and E-dominated.
10. The clone generated by a function of type majority which is hyperplanely of type min and N-dominated.

The Meta Problem

The Meta Problem

Meta-Problem of Graph-SAT(Ψ)

INPUT: A finite set Ψ of graph formulas.

QUESTION: Is Graph-SAT(Ψ) in P?

The Meta Problem

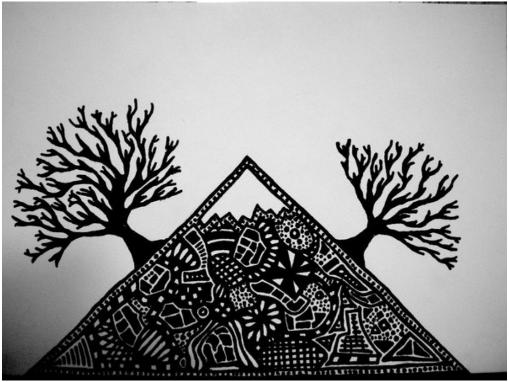
Meta-Problem of Graph-SAT(Ψ)

INPUT: A finite set Ψ of graph formulas.

QUESTION: Is Graph-SAT(Ψ) in P?

Theorem (Bodirsky + MP '10)

The Meta-Problem of Graph-SAT(Ψ) is decidable.



Part IV:

Model-complete cores / The infinite tractability conjecture

Homomorphic equivalence

Homomorphic equivalence

Only reduction method so far:

Homomorphic equivalence

Only reduction method so far:

pp interpretations = $\mathsf{HSP}^{\text{fin}}$ = clone homomorphisms

Homomorphic equivalence

Only reduction method so far:

pp interpretations = $\mathrm{HSP}^{\mathrm{fin}}$ = clone homomorphisms

Method of analysis / proving tractability:

Homomorphic equivalence

Only reduction method so far:

pp interpretations = $\mathrm{HSP}^{\mathrm{fin}}$ = clone homomorphisms

Method of analysis / proving tractability:

Canonical functions

Homomorphic equivalence

Only reduction method so far:

pp interpretations = HSP^{fin} = clone homomorphisms

Method of analysis / proving tractability:

Canonical functions

Other reduction method:

Homomorphic equivalence

Only reduction method so far:

pp interpretations = HSP^{fin} = clone homomorphisms

Method of analysis / proving tractability:

Canonical functions

Other reduction method:

Definition

τ-structures Γ, Δ are homomorphically equivalent iff Γ maps homomorphically into Δ and vice-versa.

Homomorphic equivalence

Only reduction method so far:

pp interpretations = $\mathrm{HSP}^{\mathrm{fin}}$ = clone homomorphisms

Method of analysis / proving tractability:

Canonical functions

Other reduction method:

Definition

τ-structures Γ, Δ are homomorphically equivalent iff Γ maps homomorphically into Δ and vice-versa.

Note: Homomorphically equivalent structures have equal CSPs.

Homomorphic equivalence

Only reduction method so far:

pp interpretations = $\mathrm{HSP}^{\mathrm{fin}}$ = clone homomorphisms

Method of analysis / proving tractability:

Canonical functions

Other reduction method:

Definition

τ-structures Γ, Δ are homomorphically equivalent iff
Γ maps homomorphically into Δ and vice-versa.

Note: Homomorphically equivalent structures have equal CSPs.

Could possibly obtain hard structure Δ by pp interpretations + homomorphic equivalence, but not by pp interpretations only.

Model-complete cores

Model-complete cores

Let Γ be structure with endomorphism e.

Model-complete cores

Let Γ be structure with endomorphism e.

$e[\Gamma]$ induces a "smaller" homomorphically equivalent structure in Γ.

Model-complete cores

Let Γ be structure with endomorphism e.

$e[\Gamma]$ induces a "smaller" homomorphically equivalent structure in Γ.

Idea: repeat to obtain "smallest" homomorphically equivalent structure.

Model-complete cores

Let Γ be structure with endomorphism e.

$e[\Gamma]$ induces a "smaller" homomorphically equivalent structure in Γ.

Idea: repeat to obtain "smallest" homomorphically equivalent structure.

Definition

A countable ω-categorical structure Δ is a model-complete core iff $\mathrm{Aut}(\Delta)$ is dense in $\mathrm{End}(\Delta)$.

Model-complete cores

Let Γ be structure with endomorphism e.

$e[\Gamma]$ induces a "smaller" homomorphically equivalent structure in Γ.

Idea: repeat to obtain "smallest" homomorphically equivalent structure.

Definition

A countable ω-categorical structure Δ is a model-complete core iff $\mathrm{Aut}(\Delta)$ is dense in $\mathrm{End}(\Delta)$.

Mc cores cannot be further simplified:
its endomorphisms are elementary, i.e., preserve first-order formulas.

Model-complete cores

Let Γ be structure with endomorphism e.

$e[\Gamma]$ induces a “smaller” homomorphically equivalent structure in Γ.

Idea: repeat to obtain “smallest” homomorphically equivalent structure.

Definition

A countable ω-categorical structure Δ is a model-complete core iff $\text{Aut}(\Delta)$ is dense in $\text{End}(\Delta)$.

Mc cores cannot be further simplified:
its endomorphisms are elementary, i.e., preserve first-order formulas.

Note: Property of the topological clone $\text{Pol}(\Delta)$.

Obtaining mc cores

Obtaining mc cores

Theorem (Bodirsky + Hils + Martin '10)

Every finite or ω-categorical structure is homomorphically equivalent to an ω-categorical model complete core.

Obtaining mc cores

Theorem (Bodirsky + Hils + Martin '10)

Every finite or ω-categorical structure is homomorphically equivalent to an ω-categorical model complete core.

Its mc core is unique up to isomorphism.

Obtaining mc cores

Theorem (Bodirsky + Hils + Martin '10)

Every finite or ω-categorical structure is homomorphically equivalent to an ω-categorical model complete core.

Its mc core is unique up to isomorphism.

Examples

Obtaining mc cores

Theorem (Bodirsky + Hils + Martin '10)

Every finite or ω-categorical structure is homomorphically equivalent to an ω-categorical model complete core.

Its mc core is unique up to isomorphism.

Examples

- mc core of random graph $(V; E)$: countably infinite clique

Obtaining mc cores

Theorem (Bodirsky + Hils + Martin '10)

Every finite or ω-categorical structure is homomorphically equivalent to an ω-categorical model complete core.

Its mc core is unique up to isomorphism.

Examples

- mc core of random graph $(V; E)$: countably infinite clique
- mc core of $(V; E, N)$: $(V; E, N)$

Obtaining mc cores

Theorem (Bodirsky + Hils + Martin '10)

Every finite or ω-categorical structure is homomorphically equivalent to an ω-categorical model complete core.

Its mc core is unique up to isomorphism.

Examples

- mc core of random graph $(V; E)$: countably infinite clique
- mc core of $(V; E, N)$: $(V; E, N)$
- mc core of $(\mathbb{Q}; \leq)$: one-element poset

Obtaining mc cores

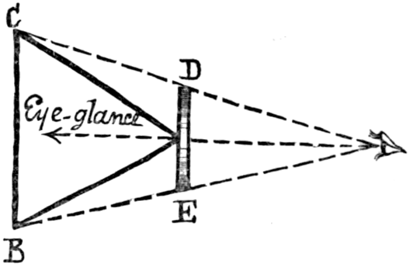
Theorem (Bodirsky + Hils + Martin '10)

Every finite or ω-categorical structure is homomorphically equivalent to an ω-categorical model complete core.

Its mc core is unique up to isomorphism.

Examples

- mc core of random graph $(V; E)$: countably infinite clique
- mc core of $(V; E, N)$: $(V; E, N)$
- mc core of $(\mathbb{Q}; \leq)$: one-element poset
- mc core of $(\mathbb{Q}; <)$: $(\mathbb{Q}; <)$



The infinite tractability conjecture

The tractability conjecture

The tractability conjecture

Fun fact: When Γ is an ω-categorical mc core and $c \in \Gamma$, then adding the relation $\{c\}$ to Γ does not increase the complexity of the CSP.

The tractability conjecture

Fun fact: When Γ is an ω-categorical mc core and $c \in \Gamma$, then adding the relation $\{c\}$ to Γ does not increase the complexity of the CSP.

Conjecture (Bulatov + Jeavons + Krokhin '05; Barto + Kozik '10)

Let Γ be finite. Let Δ be its mc core expanded by all constants. Then:

The tractability conjecture

Fun fact: When Γ is an ω-categorical mc core and $c \in \Gamma$, then adding the relation $\{c\}$ to Γ does not increase the complexity of the CSP.

Conjecture (Bulatov + Jeavons + Krokhin '05; Barto + Kozik '10)

Let Γ be finite. Let Δ be its mc core expanded by all constants. Then:

- either $\mathrm{Pol}(\Delta)$ has a homomorphism to **1** (and $\mathrm{CSP}(\Gamma)$ is NP-hard),

The tractability conjecture

Fun fact: When Γ is an ω-categorical mc core and $c \in \Gamma$, then adding the relation $\{c\}$ to Γ does not increase the complexity of the CSP.

Conjecture (Bulatov + Jeavons + Krokhin '05; Barto + Kozik '10)

Let Γ be finite. Let Δ be its mc core expanded by all constants. Then:

- either $\mathrm{Pol}(\Delta)$ has a homomorphism to **1** (and $\mathrm{CSP}(\Gamma)$ is NP-hard),
- or $\mathrm{Pol}(\Delta)$ contains a cyclic operation f of arity $n > 1$, i.e.,

$$f(x_1, \ldots, x_n) = f(x_2, \ldots, x_n, x_1)$$

and $\mathrm{CSP}(\Gamma)$ *is in P*.

Reducts of $(\mathbb{Q}; <)$

Reducts of $(\mathbb{Q}; <)$

Theorem (Bodirsky + Kara '08, reformulated)

Let Γ be a reduct of $(\mathbb{Q}; <)$. Let Δ be its mc core. Then:

Reducts of $(\mathbb{Q}; <)$

Theorem (Bodirsky + Kara '08, reformulated)

Let Γ be a reduct of $(\mathbb{Q}; <)$. Let Δ be its mc core. Then:

- either there is an expansion Δ' of Δ by finitely many constants such that $\text{Pol}(\Delta')$ has a continuous homomorphism to $\mathbf{1}$ (and $\text{CSP}(\Gamma)$ is NP-hard);

Reducts of $(\mathbb{Q}; <)$

Theorem (Bodirsky + Kara '08, reformulated)

Let Γ be a reduct of $(\mathbb{Q}; <)$. Let Δ be its mc core. Then:

- either there is an expansion Δ' of Δ by finitely many constants such that $\mathrm{Pol}(\Delta')$ has a continuous homomorphism to **1** (and $\mathrm{CSP}(\Gamma)$ is NP-hard);
- or there are $f(x_1, x_2) \in \mathrm{Pol}(\Gamma)$ and $\alpha, \beta, \gamma \in \mathrm{Aut}(\mathbb{Q}; <)$ such that

$$\alpha f(x, x, y) = \beta f(x, y, x) = \gamma f(x, y, x)$$

  and $\mathrm{CSP}(\Gamma)$ is in P.

Reducts of the random graph G

Reducts of the random graph G

Theorem (Bodirsky + MP '11, reformulated)

Let Γ be a reduct of G. Then:

- either $\mathrm{Pol}(\Gamma)$ has a continuous homomorphism to **1** (and $\mathrm{CSP}(\Gamma)$ is NP-hard);

Reducts of the random graph G

Theorem (Bodirsky + MP '11, reformulated)

Let Γ be a reduct of G. Then:

- either $\mathrm{Pol}(\Gamma)$ has a continuous homomorphism to **1** (and $\mathrm{CSP}(\Gamma)$ is NP-hard);
- or there are $f(x_1, x_2, x_3) \in \mathrm{Pol}(\Gamma)$ and $\alpha \in \mathrm{Aut}(G)$ such that

$$f(x_1, x_2, x_3) = \alpha(f(x_3, x_1, x_2))$$

  and $\mathrm{CSP}(\Gamma)$ is in P.

Proof method

Proof method

- Identify relations R such that $\mathrm{Pol}(V; R)$ has a continuous homomorphism to **1**.

Proof method

- Identify relations R such that $\mathrm{Pol}(V; R)$ has a continuous homomorphism to **1**.
- If none of those hard relations is pp definable in Γ, then there are functions in $\mathrm{Pol}(\Gamma)$ witnessing this.

Proof method

- Identify relations R such that $\mathrm{Pol}(V; R)$ has a continuous homomorphism to **1**.
- If none of those hard relations is pp definable in Γ, then there are functions in $\mathrm{Pol}(\Gamma)$ witnessing this.
- Using Ramsey theory we find canonical (='nice') such polymorphisms.

Proof method

- Identify relations R such that $\mathrm{Pol}(V; R)$ has a continuous homomorphism to **1**.
- If none of those hard relations is pp definable in Γ, then there are functions in $\mathrm{Pol}(\Gamma)$ witnessing this.
- Using Ramsey theory we find canonical (='nice') such polymorphisms.
- Canonical polymorphisms are essentially finite functions.

Proof method

- Identify relations R such that $\text{Pol}(V; R)$ has a continuous homomorphism to $\mathbf{1}$.
- If none of those hard relations is pp definable in Γ, then there are functions in $\text{Pol}(\Gamma)$ witnessing this.
- Using Ramsey theory we find canonical (='nice') such polymorphisms.
- Canonical polymorphisms are essentially finite functions.

  So they allow for combinatorial analysis and algorithmic use, and "should" satisfy equations.

Finitely bounded homogeneous structures

Finitely bounded homogeneous structures

Fact: There are homogeneous digraphs with undecidable CSP.

Finitely bounded homogeneous structures

Fact: There are homogeneous digraphs with undecidable CSP.

Definition

A structure with finite relational signature is finitely bounded iff its age is determined by finitely many forbidden substructures.

Finitely bounded homogeneous structures

Fact: There are homogeneous digraphs with undecidable CSP.

Definition

A structure with finite relational signature is finitely bounded iff its age is determined by finitely many forbidden substructures.

Examples: $(\mathbb{Q}; <)$ and the random graph.

Finitely bounded homogeneous structures

Fact: There are homogeneous digraphs with undecidable CSP.

Definition

A structure with finite relational signature is finitely bounded iff its age is determined by finitely many forbidden substructures.

Examples: $(\mathbb{Q}; <)$ and the random graph.

Fact: The CSP of any reduct of a finitely bounded structure is in NP.

Infinite tractability conjecture

Infinite tractability conjecture

Let Γ be a reduct of a finitely bounded homogeneous structure.

Infinite tractability conjecture

Let Γ be a reduct of a finitely bounded homogeneous structure.

Let Δ be its model-complete core.

Infinite tractability conjecture

Let Γ be a reduct of a finitely bounded homogeneous structure.

Let Δ be its model-complete core.

Conjecture (Bodirsky + MP '13)

- either there is an expansion Δ' of Δ by finitely many constants such that $\mathrm{Pol}(\Delta')$ has a continuous homomorphism to $\mathbf{1}$ (and $\mathrm{CSP}(\Gamma)$ is NP-hard);

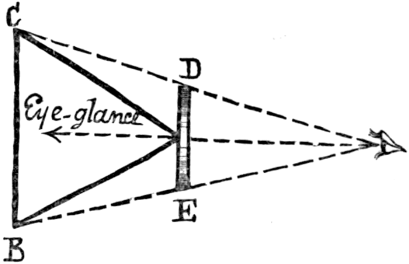
Infinite tractability conjecture

Let Γ be a reduct of a finitely bounded homogeneous structure.

Let Δ be its model-complete core.

Conjecture (Bodirsky + MP '13)

- either there is an expansion Δ' of Δ by finitely many constants such that $\mathrm{Pol}(\Delta')$ has a continuous homomorphism to $\mathbf{1}$ (and $\mathrm{CSP}(\Gamma)$ is NP-hard);
- $\mathrm{Pol}(\Delta)$ satisfies a non-trivial equation, and $\mathrm{CSP}(\Gamma)$ is tractable.



Future work

Tools

Tools

- pp preservation
- continuous clone homomorphism / Topological Birkhoff
- canonical functions
- model-complete cores

Future work

Future work

- Does every homogeneous structure in a finite relational language have a homogeneous Ramsey expansion by finitely many relation symbols?

  (Bodirsky + MP + Tsankov, *Decidability of definability*)

Future work

- Does every homogeneous structure in a finite relational language have a homogeneous Ramsey expansion by finitely many relation symbols?

  (Bodirsky + MP + Tsankov, *Decidability of definability*)

- If Pol(Γ) has a homomorphism to **1**, does it also have a continuous homomorphism?

  (Bodirsky + MP + Pongrácz, *Projective clone homomorphisms*)

Future work

- Does every homogeneous structure in a finite relational language have a homogeneous Ramsey expansion by finitely many relation symbols?

  (Bodirsky + MP + Tsankov, *Decidability of definability*)

- If Pol(Γ) has a homomorphism to **1**, does it also have a continuous homomorphism?

  (Bodirsky + MP + Pongrácz, *Projective clone homomorphisms*)

- Clarify relationship between canonical functions and their finite counterparts (algorithmic / equational).

Distress not yourself if you cannot at first understand the deeper mysteries of Spaceland.
By degrees they will dawn upon you.