Asymptotics for Causal Linear Fields

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Introduction
Stationary linear processes and fields
Gordin's Method
Empirical CLT for a linear field
Gordin's Method in 2 Dimensions
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Joint work with Neville Weber, University of Sydney

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Introduction

A stationary process $(X_i : i \in \mathbf{Z})$ or stationary random field $(X_{i,j} : i, j \in \mathbf{Z})$ has short memory (is short-range dependent) if and only if its covariance function is absolutely summable:

$$\sum_i |Cov(X_0 X_i)| < \infty, \text{ resp., } \sum_i \sum_j |Cov(X_{0,0} X_{i,j})| < \infty$$

Most known asymptotic results for short memory fields involve conditions on mixing or association. These are not needed in the case of a causal linear process (field).

Stationary linear processes and fields

Let $(\xi_i : i \in \mathbf{Z})$ be i.i.d. with mean 0 and variance 1 and let $(a_i : i \in \mathbf{Z})$ be square summable: $\sum_i a_i^2 < \infty$.

$$X_s := \sum_i a_i \xi_{s-i}$$

is a linear process.

Let $(\xi_{i,j} : i,j \in \mathbf{Z})$ be i.i.d. with mean 0 and variance 1 and let $(a_{i,j} : i,j \in \mathbf{Z})$ be square summable: $\sum_i \sum_j a_{i,j}^2 < \infty$.

$$X_{s,t} := \sum_i \sum_j a_{i,j} \xi_{s-i,t-j}$$

is a linear field.

Stationary linear processes and fields

- The linear process (field) has short memory if and only if $\sum_i |a_i| < \infty$ (resp., $\sum_i \sum_j |a_{i,j}| < \infty$).
- The linear process (field) is *causal* if $a_i = 0\ \forall i < 0$ (resp., $a_{i,j} = 0$ if $i < 0$ or $j < 0$) i.e.

$$X_s := \sum_{i=0}^{\infty} a_i \xi_{s-i}$$

$$X_{s,t} := \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{i,j} \xi_{s-i,t-j}$$

- Causal linear fields have received attention in the recent economics literature.

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Stationary linear processes and fields

Let F denote the marginal distribution of X_i (resp., $X_{i,j}$). We will consider the asymptotic behaviour of the corresponding empirical distributions:

$$F_n(x) := \frac{1}{n} \sum_{i=1}^{n} I(X_i \leq x)$$

$$F_{m,n}(x) := \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} I(X_{i,j} \leq x)$$

.

Stationary linear processes and fields

Theorem

(Doukhan and Surgailis (1998)) Let (X_i) be a short memory linear process and assume that the ξ_i's have a bounded density. For each $x \in \mathbf{R}$

$$\sqrt{n}(F_n(x) - F(x)) \xrightarrow{\mathcal{D}} W(x)$$

where $W(x) \sim N(0, \sigma^2(x))$ and

$$\sigma^2(x) := \sum_{i=-\infty}^{\infty} Cov(I(X_0 \leq x), I(X_i \leq x)).$$

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Stationary linear processes and fields

If $E[\xi_0^4] < \infty$, *then*

$$\sqrt{n}(F_n(\cdot) - F(\cdot)) \stackrel{\mathcal{D}}{\to} W$$

in $D(\mathbf{R})$ *where* W *is a centered Gaussian process with covariance function*

$$\sigma(x,y) := \sum_{i=-\infty}^{\infty} Cov(I(X_0 \le x), I(X_i \le y))).$$

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Gordin's Method (Gordin (1969))

Let $\mathcal{F}_i = \sigma\{\xi_u : u \leq i\}$. By causality, X_i is $\mathcal{F}_i$-measurable.

$$U_i(h) := P[X_i \leq x|\mathcal{F}_{i-h}] - P[X_i \leq x|\mathcal{F}_{i-h-1}]$$

are martingale differences.

- $U_i(h)$ is $\mathcal{F}_{i-h}$-measurable.

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$$U_i(h) := P[X_i \leq x | \mathcal{F}_{i-h}] - P[X_i \leq x | \mathcal{F}_{i-h-1}]$$

are martingale differences.

- $U_i(h)$ is $\mathcal{F}_{i-h}$-measurable.
- For h fixed, $U_i(h)$ is stationary in i.

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$$U_i(h) := P[X_i \leq x | \mathcal{F}_{i-h}] - P[X_i \leq x | \mathcal{F}_{i-h-1}]$$

are martingale differences.

- $U_i(h)$ is $\mathcal{F}_{i-h}$-measurable.
- For h fixed, $U_i(h)$ is stationary in i.
- $\cap_i \mathcal{F}_i = \mathcal{T}$ satisfies the 0-1 law.

Gordin's Method
Proof of the CLT

For $K < \infty$,

$$R_i^K(x) := \sum_{h=0}^{K} U_i(h) = I(X_i \leq x) - P[X_i \leq x | \mathcal{F}_{i-K-1}].$$

1. $R_i^K(x) \stackrel{a.s}{\to} R_i(x) := I(X_i \leq x) - F(x)$ as $K \to \infty$.

Gordin's Method
Proof of the CLT

For $K < \infty$,

$$R_i^K(x) := \sum_{h=0}^{K} U_i(h) = I(X_i \leq x) - P[X_i \leq x | \mathcal{F}_{i-K-1}].$$

1. $R_i^K(x) \stackrel{a.s}{\to} R_i(x) := I(X_i \leq x) - F(x)$ as $K \to \infty$.
2. $\sqrt{n}(F_n(x) - F(x)) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} R_i(x) \approx \frac{1}{\sqrt{n}} \sum_{i=1}^{n} R_i^K(x)$.

Gordin's Method
Proof of the CLT

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n} R_i^K(x) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sum_{h=0}^{K} U_i(h)$$
$$\approx \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sum_{h=0}^{K} U_{i+h}(h)$$
$$= \frac{1}{\sqrt{n}}\sum_{i=1}^{n} M_i^K$$

$\{M_i^K := \sum_{h=0}^{K} U_{i+h}(h) : i \geq 1\}$ is a *stationary and ergodic* sequence of martingale differences.

Gordin's Method

Proof of the CLT

$$\begin{aligned} E[(M_0^K)^2] &= \sum_{i=-\infty}^{\infty} Cov(R_0^K(x), R_i^K(x)) \\ &:= \sigma_K^2(x) \xrightarrow{K} \sigma^2(x). \end{aligned}$$

CLT for stationary ergodic martingale differences $\Rightarrow$

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} R_i^K(x) \xrightarrow{\mathcal{D}} W_K(x)$$

where $W_K(x) \sim N(0, \sigma_K^2(x))$. Now let $K \to \infty$.

Fourth moments are used to prove tightness. $\square$

Empirical CLT for a linear field

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Theorem

(Ivanoff and Weber (2010)) Let $(X_{i,j})$ be a short memory linear field and assume that the $\xi'_{i,j}$s have a bounded density. For each $x \in \mathbf{R}$

$$\sqrt{mn}(F_{m,n}(x) - F(x)) \xrightarrow{\mathcal{D}} W(x)$$

as $m, n \to \infty$ where $W(x) \sim N(0, \sigma^2(x))$ and

$$\sigma^2(x) := \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} Cov(I(X_{0,0} \leq x), I(X_{i,j} \leq x))).$$

Empirical CLT for a linear field

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If $E[\xi_{0,0}^4] < \infty$*, then as* $m, n \to \infty$

$$\sqrt{mn}(F_{m,n}(\cdot) - F(\cdot)) \xrightarrow{\mathcal{D}} W$$

in $D(\mathbf{R})$ *where* W *is a centered Gaussian process with covariance function* $\sigma(x, y) :=$

$$\sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} Cov(I(X_{0,0} \le x), I(X_{i,j} \le y))).$$

Gordin's Method in 2 Dimensions

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The approach: find a *total* order $\prec$ on the lattice that allows us to create a (one-dimensional) martingale difference sequence that has some sort of stationarity and ergodicity properties.

$\prec$ must be consistent with the usual *partial* order $\leq$ on $\mathbf{Z}^2$ ($i \leq i'$ and $j \leq j'$ $\Leftrightarrow (i,j) \leq (i',j') \Rightarrow (i,j) \preceq (i',j')$)

Using $\prec$, we must be able to

- to count backward on $(-\infty, i] \times (-\infty, j]$ to $(-\infty, -\infty)$
- to count forward on $\mathbf{Z}^2_+$ to (∞, ∞)

Gordin's Method in 2 Dimensions

Solution: We use the *diagonal* order:

$$(i,j) \prec (i',j') \quad \Leftrightarrow \quad i+j < i'+j'$$
$$\underline{\text{or}} \quad i+j = i'+j' \text{ and } \; i < i'.$$

- $(i,j) \leq (i',j') \Rightarrow (i,j) \preceq (i',j')$ (consistency)
- In the diagonal order $\preceq$, for the immediate predecessor of (i,j) is $(i-1, j+1)$.
- However, for $r \leq i$,

$$\sup\{(h,k) : (h,k) \prec (r,j), (h,k) \leq (i,j)\} = (i, r+j-i-1)$$

Gordin's Method in 2 Dimensions

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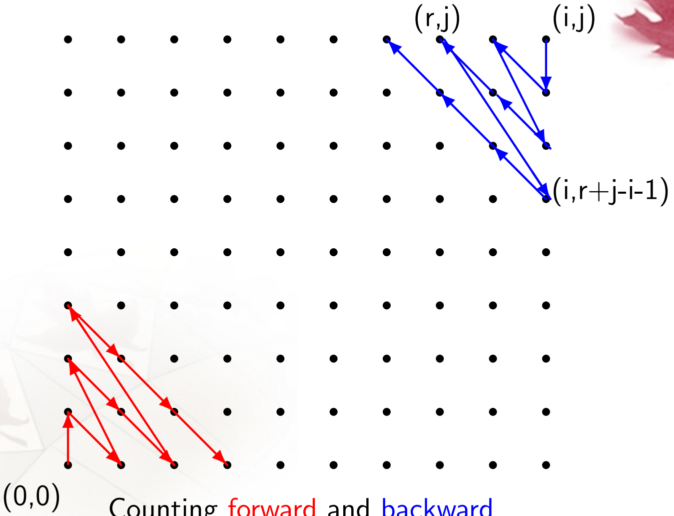

Counting forward and backward

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Gordin's Method in 2 Dimensions

Remember: the $\xi'_{i,j}s$ are i.i.d.
Define

$$\begin{aligned} \mathcal{F}_{i,j} &= \sigma\{\xi_{u,v} : \ (u,v) \leq (i,j)\} \\ \mathcal{G}_{i,j} &= \sigma\{\xi_{u,v} : \ (u,v) \preceq (i,j)\} \\ \mathcal{G}^D_\ell &= \sigma\{\xi_{u,v} : u+v \leq \ell\}. \end{aligned}$$

- By causality $X_{i,j}$ is $\mathcal{F}_{i,j}$-measurable.
- Since $\preceq$ is consistent with $\leq$, $\mathcal{F}_{i,j} \subseteq \mathcal{G}_{i,j}$.

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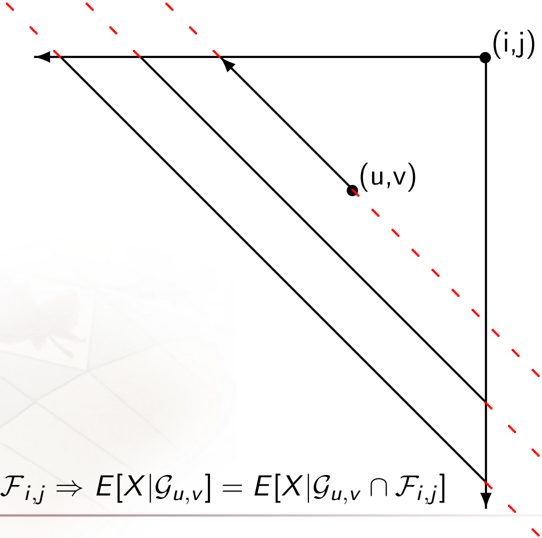

$$X \in \mathcal{F}_{i,j} \Rightarrow E[X|\mathcal{G}_{u,v}] = E[X|\mathcal{G}_{u,v} \cap \mathcal{F}_{i,j}]$$

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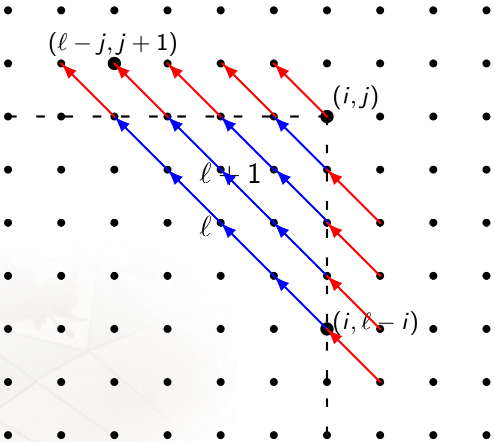

$$E[X|\mathcal{G}_{\ell-j,j+1}] = E[X|\mathcal{G}_{\ell}^{D}] = E[X|\mathcal{G}_{i,\ell-i}]$$

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Gordin's Method in 2 Dimensions

Some observations:

If X is $\mathcal{F}_{i,j}$ measurable then by independence of the $\xi_{i,j}$'s,

1. if $(u,v) \preceq (i,j)$

$$
\begin{aligned}
E[X|\mathcal{G}_{u,v}] &= E[X|\xi_{h,\ell};\ (h,\ell) \preceq (u,v)] \\
&= E[X|\xi_{h,\ell};\ (h,\ell) \preceq (u,v),\ (h,\ell) \leq (i,j)] \\
&= E[X|\mathcal{G}_{u,v} \cap\ \mathcal{F}_{i,j}].
\end{aligned}
$$

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If X is $\mathcal{F}_{i,j}$ measurable then by independence of the $\xi_{i,j}$'s,

1. if $(u,v) \preceq (i,j)$

$$\begin{aligned} E[X|\mathcal{G}_{u,v}] &= E[X|\xi_{h,\ell};\ (h,\ell) \preceq (u,v)] \\ &= E[X|\xi_{h,\ell};\ (h,\ell) \preceq (u,v),\ (h,\ell) \leq (i,j)] \\ &= E[X|\mathcal{G}_{u,v} \cap\ \mathcal{F}_{i,j}]. \end{aligned}$$

2. if $\ell < i + j$, then

$$E[X|\mathcal{G}_{\ell-j,j+1}] = E[X|\mathcal{G}^D_\ell] = E[X|\mathcal{G}^D_\ell \cap \mathcal{F}_{i,j}] = E[X|\mathcal{G}_{i,\ell-i}]$$

Gordin's Method in 2 Dimensions
Proof of the Empirical CLT

Recall:

$$X_{i,j} := \sum_{u=0}^{\infty}\sum_{v=0}^{\infty} a_{u,v}\xi_{i-u,j-v}$$

For all $(i,j) \in \mathbf{Z}^2$ and $h, k \geq 0$ define

$$U_{i,j}(h,k) := P[X_{i,j} \leq x|\mathcal{G}_{i-h,j-k}] - P[X_{i,j} \leq x|\mathcal{G}_{i-h-1,j-k+1}],$$

noting that if $k = 0$,

$$\begin{aligned} U_{i,j}(h,0) &= P[X_{i,j} \leq x|\mathcal{G}_{i-h,j}] - P[X_{i,j} \leq x|\mathcal{G}_{i-h-1,j+1}] \\ &= P[X_{i,j} \leq x|\mathcal{G}_{i-h,j}] - P[X_{i,j} \leq x|\mathcal{G}_{i,j-h-1}] \end{aligned}$$

Gordin's Method in 2 Dimensions

Proof of the Empirical CLT

- For h, k fixed, the $U_{i,j}(h,k)$ are stationary in i, j.

Gordin's Method in 2 Dimensions
Proof of the Empirical CLT

- For h, k fixed, the $U_{i,j}(h,k)$ are stationary in i, j.
- $U_{i,j}(h,k)$ is $\mathcal{G}_{i-h,j-k}$-measurable.

Gordin's Method in 2 Dimensions
Proof of the Empirical CLT

- For h, k fixed, the $U_{i,j}(h,k)$ are stationary in i, j.
- $U_{i,j}(h,k)$ is $\mathcal{G}_{i-h,j-k}$-measurable.
- $\cap \mathcal{G}_{\ell}^{D} = \mathcal{T}$ satisfies the 0-1 law .

Gordin's Method in 2 Dimensions
Proof of the Empirical CLT

Recall

$$F_{m,n}(x) := \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} I(X_{i,j} \le x)$$

Condition backwards in $\prec$ from (i,j) over K diagonals:

$$\begin{aligned} R_{i,j}^{K}(x) &:= \sum_{\ell=0}^{K}\sum_{h=0}^{\ell} U_{i,j}(h, \ell-h) \\ &= I(X_{i,j} \le x) - P(X_{i,j} \le x | \mathcal{G}_{i+j-(K+1)}^{D}) \end{aligned}$$

Gordin's Method in 2 Dimensions

Proof of the Empirical CLT

- $R^K_{i,j}(x) \overset{a.s}{\to} R_{i,j}(x) := I(X_i \leq x) - F(x)$ as $K \to \infty$.

Gordin's Method in 2 Dimensions

Proof of the Empirical CLT

- $R_{i,j}^K(x) \overset{a.s}{\to} R_{i,j}(x) := I(X_i \le x) - F(x)$ as $K \to \infty$.
-

$$\begin{aligned} \sqrt{mn}(F_{m,n}(x) - F(x)) &= \frac{1}{\sqrt{mn}} \sum_{i=1}^{m} \sum_{j=1}^{n} R_{i,j}(x) \\ &\approx \frac{1}{\sqrt{mn}} \sum_{i=1}^{m} \sum_{j=1}^{n} R_{i,j}^K(x) \end{aligned}$$

Gordin's Method in 2 Dimensions
Proof of the Empirical CLT

$\frac{1}{\sqrt{mn}} \sum_{i=1}^{m} \sum_{j=1}^{n} R_{i,j}^{K}(x)$

$$= \frac{1}{\sqrt{mn}} \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{\ell=0}^{K} \sum_{h=0}^{\ell} U_{i,j}(h, \ell - h)$$

$$\approx \frac{1}{\sqrt{mn}} \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{\ell=0}^{K} \sum_{h=0}^{\ell} U_{i+h,j+\ell-h}(h, \ell - h)$$

$$= \frac{1}{\sqrt{mn}} \sum_{i=1}^{m} \sum_{j=1}^{n} M_{i,j}^{K}$$

where $M_{i,j}^{K} = \sum_{\ell=0}^{K} \sum_{h=0}^{\ell} U_{i+h,j+\ell-h}(h, \ell - h)$.

Gordin's Method in 2 Dimensions

Proof of the Empirical CLT

$\{M_{ij}^K : 1 \leq i \leq m, 1 \leq j \leq n\}$ is

- a stationary ergodic array (in two dimensions)

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Gordin's Method in 2 Dimensions
Proof of the Empirical CLT

$\{M_{ij}^K : 1 \leq i \leq m, 1 \leq j \leq n\}$ is

- a stationary ergodic array (in two dimensions)
- a sequence of identically distributed (but not stationary) mn martingale differences in the total order $\preceq$ (in one dimension).

Gordin's Method in 2 Dimensions
Proof of the Empirical CLT

A martingale CLT due to McLeish (1974) can now be applied.

- Since $|U_{i,j}(h,k)| \leq 1$ and $M_{i,j}^K$ contains $\frac{(K+1)(K+2)}{2}$ summands,
$$\max_{1\leq i\leq m, 1\leq j\leq n} \frac{1}{\sqrt{mn}}|M_{i,j}^K|$$
is uniformly bounded and converges to 0 a.s.
- The two dimensional ergodic theorem implies that
$$\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}(M_{i,j}^K)^2 \xrightarrow{P} E((M_{0,0}^K)^2).$$

Gordin's Method in 2 Dimensions

Proof of the Empirical CLT

$$
\begin{aligned}
E((M_{0,0}^K)^2) &= \sum_{i \in \mathbf{Z}} \sum_{j \in \mathbf{Z}} Cov(R_{0,0}^K(x) R_{i,j}^K(x)) \\
&= \sigma_K^2(x) \to \sigma^2(x) \text{ as } K \to \infty
\end{aligned}
$$

The pointwise (and fidi) CLT follows assuming only second moments.

Fourth moments are used to prove tightness. □

Conclusion

Gordin's martingale method can be extended to two dimensions via the diagonal order if

- the underlying filtration is generated by i.i.d. random variables, and
- the stationary process is causal

The technique combines one-dimensional martingale methods with two-dimensional ergodicity.

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