Inference of Weighted V-statistics for Non-stationary Time Series And Its Applications

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Statistics to be investigated

Let $\{X_{n,i}\}_{i=1}^{n}$ be an array of non-stationary time series. Consider the statistics

$$V_n = \sum_{k=1}^{n} \sum_{j=1}^{n} W_n(t_k, t_j) H(X_{n,k}, X_{n,j}),$$

where $t_k = k/n$, $k = 1, 2, \cdots, n$ are the rescaled times.

Example: nonparametric estimation of non-stationary time series.

$F(t, \cdot)$: Marginal CDF of $\{X_j\}$ at rescaled time t, $t \in [0, 1]$.
Consider estimating

$$\theta(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(x, y)\, dF(t, x) dF(t, y). \tag{1}$$

For example, $H(x, y) = (x - y)^2/2$ implies $\theta(t)$ is the marginal variance function of $\{X_j\}$.

Example: nonparametric estimation of non-stationary time series.

$\theta(t)$ can be estimated via

$$(\hat{\theta}_{b_n}(t), \hat{\eta}_1, \hat{\eta}_2) = \underset{(\eta_0, \eta_1 \eta_2) \in \mathbb{R}}{\operatorname{argmin}} \sum_{j,k=1}^{n} (H(X_j, X_k) - \eta_0 - \eta_1(t_j - t) - \eta_2(t_k - t))^2 W_n(t_j, t_k). \quad (2)$$

$$W_n(t_j, t_k) = K((t_j - t)/b_n, (t_k - t)/b_n)/(nb_n),$$

and b_n is bandwidth. Then the investigation of $\hat{\theta}_{b_n}(t)$ is equivalent to that of

$$V_n = \sum_{k=1}^{n} \sum_{j=1}^{n} W_n(t_k, t_j) H(X_k, X_j).$$

Example: nonparametric specification of non-stationary time series.

For a fixed t, Let $\phi(t,x) = \int_{\mathbb{R}} G(x,y)\,dF(t,y)$. E.g., $G(x,y) = I(y \le x)$, or e^{iyx}.

$H_0 : \phi(t,x) = \phi_0(t,x) \longleftrightarrow H_a : \phi(t,x) \ne \phi_0(t,x), \quad t \in [0,1], x \in \mathbb{R},$

where $\phi_0(\cdot,\cdot)$ is a known function. $\phi(t,x)$ can be estimated by the following

$$\hat{\phi}(t,x) = \frac{1}{nb_n} \sum_{j=1}^{n} \alpha_j(x) K_{b_n}(t_j - t),$$

where $\alpha_j(x) = G(x, X_j)$. $\mathcal{L}^2$ type test statistic:

$$L_n = \int_{b_n}^{1-b_n} \int_{\mathbb{R}} |\hat{\phi}(t,x) - \phi_0(t,x)|^2 w(x)\,dxdt,$$

where $w(x)$ a user chosen non-negative and smooth weight function, $\int_{\mathbb{R}} (1 + |x|^3) w(x)\,dx < \infty$.

Example: nonparametric specification of non-stationary time series.

Under H_0, the investigation of L_n boils down to the investigation of

$$V_n = \sum_{k=1}^{n} \sum_{j=1}^{n} W_n(t_k, t_j) H(X_k, X_j),$$

where $H(a, b) = \int_{\mathbb{R}^2} G(x, a) G(x, b) w(x)\, dx$ and
$W_n(t_k, t_j) = \int_0^1 K_{b_n}(t_j - t) K_{b_n}(t_k - t)\, dt / (nb_n)$.

Example: spectral analysis of non-stationary time series.

Define the spectral density of a centered time series $\{X_j\}$ at time t as

$$f(t,\lambda) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\gamma(t,k)\cos(k\lambda), \quad \lambda \in [0, 2\pi],$$

where $\gamma(t,k$ is the k-th order auto covariance of $\{X_j\}$ at time t.
Classic smoothed periodogram estimate of the spectral density

$$\tilde{f}_n(\lambda) = \int_{\mathbb{R}} \frac{1}{m_n} K\left(\frac{u}{m_n}\right) I_n(\lambda + 2\pi u/n)\, du,$$

where $I_n(\lambda)$ is the periodogram of the series at frequency λ and m_n is a blocksize.
$\tilde{f}_n(\lambda)$ is expected to be consistent for $\int_0^1 f(t,\lambda)\,dt$ under non-stationarity.

Non-stationary time series models

- Class I: $X_j = V_j(\mathcal{F}_j)$, where $\mathcal{F}_j = (\cdots, \varepsilon_0, \cdots, \varepsilon_{j-1}, \varepsilon_j)$ and ε_j's are i.i.d. random variables.

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- Class II:

Definition

We say $\{X_i\}_{i=1}^n$ is piecewise locally stationary with r break points (PLS(r)) if there exist constants $0 = b_0 < b_1 < \cdots < b_r < b_{r+1} = 1$ and nonlinear filters $G_0, G_1, \cdots, G_r$, such that

$$X_i = G_j(t_i, \mathcal{F}_i), \text{ if } b_j < t_i \leq b_{j+1}. \quad (3)$$

Fourier transforms

Suppose that $H^*(x, y) = H(x, y)/(L(x)L(y)) \in L^1\{\mathbb{R}^2\}$.

$$H^*(x, y) = \int_{\mathbb{R}^2} g(u, v)e^{i(xu+yv)}\, dudv.$$

Then

$$V_n = \int_{\mathbb{R}^2} g(x, y) \sum_{k,j=1}^{n} W_n(t_k, t_j)\beta_k(x)\beta_j(y)\, dxdy, \tag{4}$$

where $\beta_k(x) = L(X_k)\exp(ixX_k)$.

The Hoeffding decomposition

Define

$$H_j(x) = \mathbb{E}[H(x, X_j)] = \mathbb{E}[H(X_j, x)] = \int_{\mathbb{R}^2} g(t,s)L(x)\exp(itx)\mathbb{E}[\beta_j(s)]\,dtds$$

and $\gamma_j(x) = \beta_j(x) - \mathbb{E}[\beta_j(x)]$. Then

$$V_n - \mathbb{E}V_n = 2N_n + D_n - \mathbb{E}[D_n], \text{ where} \tag{5}$$

$$\begin{aligned} N_n &= \sum_{k=1}^{n}\sum_{j=1}^{n} W_n(t_k, t_j)\{H_j(X_k) - \mathbb{E}[H_j(X_k)]\}, \\ D_n &= \int_{\mathbb{R}^2} g(x,y)\sum_{k,j} W_n(t_k,t_j)\gamma_k(x)\gamma_j(y)\,dxdy. \end{aligned}$$

Non-degenerate V-statistics: CLT of N_n.

(A1) For some $\eta \in (0,1]$, $H^*(t,s) := H(t,s)/[L(t)L(s)] \in W_0^\eta(\mathbb{R}^2)$ for some function $L(\cdot)$.

(A2) Define $W_{j,\cdot} = \sum_{r=1}^n |W_n(t_j,t_r)|$, $j = 1,2,\cdots,n$. Let $W_{j,\cdot} = 0$ for $j > n$. Let $W^{(n)} = \sum_{j=1}^n W_{j,\cdot}^2$. For sequences l_n, m_n and $s_n = l_n + m_n$, define

$$A_j = \sum_{k=1}^{l_n} W^2_{s_n(j-1)+k,\cdot} \text{ and } a_j = \sum_{k=l_n+1}^{s_n} W^2_{s_n(j-1)+k,\cdot}, j = 1,2,\cdots \lceil n/s_n \rceil.$$

Assume that there exist sequences $m_n/\log n \to \infty$ with $l_n/n \to 0$ and $m_n/l_n \to 0$, such that

$$\sum_j a_j/W^{(n)} \to 0 \text{ and } \max_j A_j/W^{(n)} \to 0.$$

(A3) $\phi_n := \mathrm{Var}(N_n)/W^{(n)} \geq c$ for some $c > 0$ and sufficiently large n.

(A4) The dependence measures $\delta_X(k, 4+2\epsilon) = O(\rho^k)$ for some $\rho \in [0,1)$ and $\epsilon > 0$.

Non-degenerate V-statistics: CLT of N_n.

Theorem

Under conditions (A1)-(A5), and some additional mild moment conditions, we have, for Class I time series $\{X_j\}$,

$$N_n/\sqrt{Var(N_n)} \Rightarrow N(0,1).$$

Note that $\text{Var}(N_n) \propto W^{(n)}$.

Non-degenerate V-statistics: order of D_n.

(A6) Assume that there exist functions f_n and a $p \in (0, \infty)$, such that

$$|W_n(t_l, t_m) - W_n(t_k, t_j)| \le f_n(t_k, t_j)|(l, m) - (k, j)|^p$$

for all integers $k, j, l, m \in \{1, 2, \cdots, n\}$.

Theorem

Assume that conditions (A4) hold with $4 + 2\epsilon$ therein replaced by 8. Further assume (A1) and (A6). Then

$$\|D_n - \mathbb{E}D_n\|^2 = O(W_{(n)} + \Delta_n), \text{ where } W_{(n)} = \sum_{k=1}^{n}\sum_{j=1}^{n} W_n^2(t_k, t_j)$$

and $\Delta_n = \sum_{k=1}^{n}\sum_{j=1}^{n} |W_n(t_k, t_j)| f_n(t_k, t_j)$.

Degenerate V-statistics.

Without loss of generality, assume that $0 < c \leq \sum_{k=1}^{n} \sum_{j=1}^{n} W_n^2(t_k, t_j) \leq C < \infty$ in the study of degenerate V-statistics.

Theorem

Under regularity conditions similar to (A1)-(A6) and local stationarity conditions, there exist constants $\alpha_{n,1}, \alpha_{n,2}, \cdots$ with $\sum_{k=1}^{\infty} \alpha_{n,k}^2 = O(1)$ and i.i.d. standard normal random variables $Z_1, Z_2, \cdots$, such that, for class II time series $\{X_j\}$,

$$\pi(law(D_n - \mathbb{E}D_n), law(\sum_{j=1}^{\infty} \alpha_{n,j}(Z_j^2 - 1))) \to 0,$$

where $\pi(\cdot, \cdot)$ is the Lévy-Prokhorov metric.

Degenerate V-statistics.

Theorem

Let $\theta_{n,1}, \theta_{n,2}, \cdots, \theta_{n,n}$ be the eigenvalues of the matrix $\{W_n(t_j, t_k)\}_{j,k=1,\cdots,n}$ with $|\theta_{n,1}| \geq |\theta_{n,2}| \geq |\theta_{n,n}|$. Assume that $\theta_{n,1} \to 0$. Then under the conditions of Theorem 2, we have for any bounded and continuous function $h(\cdot)$

$$|\mathbb{E}h[D_n - \mathbb{E}D_n] - \mathbb{E}h\{N(0, Var[D_n])\}| \to 0. \quad (6)$$

Degenerate V-statistics.

Theorem

Suppose that (a): $W_n(t,s) = Q_2((t-a)/b_n,(s-b)/b_n)/(nb_n)$ *for some well-behaved* Q_2*; or (b):*

$$W_n(t,s) = \sum_{j=1}^{\infty} a_j g_{1,j}((t-a)/b_n) g_{2,j}((s-b)/b_n)/(nb_n)$$

, where $\sum_{j=1}^{\infty} |a_j| < \infty$*,* $b_n \to 0$ *and* $g_{1,j}(\cdot)$ *and* $g_{2,j}(\cdot)$ *are continuous functions on* $\mathbb{R}$ *with support* $[-1,1]$*. Then under the conditions of Theorem 2, there exist constants* $\alpha_1, \alpha_2, \cdots$ *with* $\sum_{j=1}^{\infty} \alpha_j^2 < \infty$ *and i.i.d. standard normal random variables* $Z_1, Z_2, \cdots$*, such that*

$$D_n - \mathbb{E}D_n \Rightarrow \sum_{j=1}^{\infty} \alpha_j (Z_j^2 - 1).$$

Nonparametric estimation of non-stationary time series.

Theorem

Assume that conditions of Theorem 1 hold. Then we have

$$\frac{\sqrt{nb_n}}{\sqrt{4\phi_n \int_{-1}^{1} K^2(x)\,dx}}[\hat{\theta}_{b_n}(t^*) - \theta(t^*) - B_n(t^*)] \Rightarrow N(0,1), \tag{7}$$

where $B_n(t^*) = b_n^2 \frac{\partial^2 \theta(t^*,t^*)}{\partial t^2} \int_{-1}^{1} x^2 K(x)\,dx$.

Nonparametric specification of non-stationary time series.

Consider the case $G(x,y) = e^{iyx}$, i.e.,specifying the time-varying characteristic functions. Let $K^*(\cdot) = K \star K(\cdot)$. Define

$$U_1 = \sum_{j,k=1}^{n} K^*_{b_n}(t_j - t_k) \int_{\mathbb{R}} [\cos(xX_j) - \mathbb{E}\cos(xX_j)][\cos(xX_k) - \mathbb{E}\cos(xX_k)] w(x)\, dx,$$

$$U_2 = \sum_{j,k=1}^{n} K^*_{b_n}(t_j - t_k) \int_{\mathbb{R}} [\sin(xX_j) - \mathbb{E}\sin(xX_j)][\sin(xX_k) - \mathbb{E}\sin(xX_k)] w(x)\, dx.$$

Theorem

Assume $nb_n^4/\log^4 n \to \infty$ and $nb_n^{9/2} \to 0$ and conditions of Theorem 2. Then we have under H_0 that

$$n\sqrt{b_n}L_n - \frac{C_1}{\sqrt{b_n}} \Rightarrow N(0, C_2),$$

where $C_1 = \lim_{n\to\infty} \mathbb{E}(U_1 - U_2)/n$ and $C_2 = \lim_{n\to\infty} Var[U_1 - U_2]/(n^2 b_n)$ are both finite constants.

Spectral analysis.

Let $\gamma(t,k)$ be the k-th order ACVF of $\{X_j\}$ at time t. Suppose that $\gamma(t,k)$ is $\mathcal{C}^p$ in t for each k.

Theorem

Assume that K is even. Then under the conditions of Theorem 2 and the assumption that $m \to \infty$ with $m/(n^{p/[4(p+2)]}\log^2 n) \to 0$, we have (i): if $0 < \lambda < \pi$, then

$$\sqrt{m}(\tilde{f}_n(\lambda) - \mathbb{E}\tilde{f}_n(\lambda)) \Rightarrow N(0, \int_{-1}^{1}[K(t)]^2\,dt \int_0^1 f^2(t,\lambda)\,dt);$$

and (ii): if $\lambda = 0$ or π, then
$\sqrt{m}(\tilde{f}_n(\lambda) - \mathbb{E}\tilde{f}_n(\lambda)) \Rightarrow N(0, 2\int_{-1}^{1}[K(t)]^2\,dt \int_0^1 f^2(t,\lambda)\,dt)$.

REFERENCES

Zhou, Z. (2014). Inference of weighted V-statistics for non-stationary time series and its applications. *The Annals of Statistics*, **42** 87–114.

Thank you!