# The harmonic measure of balls in random trees

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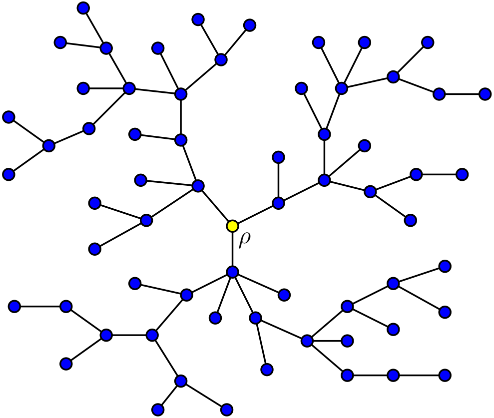
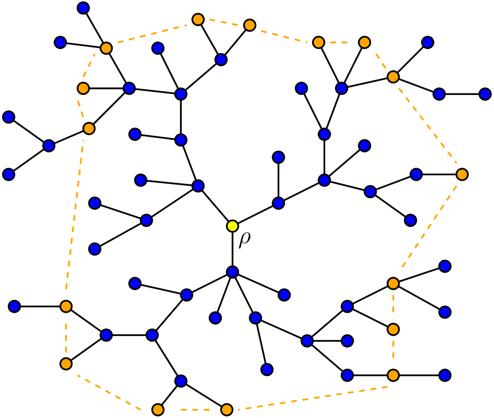
Probability on Trees and Planar Graphs, Banff 2014

# 1. Introduction: combinatorial trees




Consider a random tree with $n$ edges, with a distinguished vertex $\rho$, chosen uniformly at random from a combinatorial class

(e.g. plane trees, Cayley trees, binary trees, ...)

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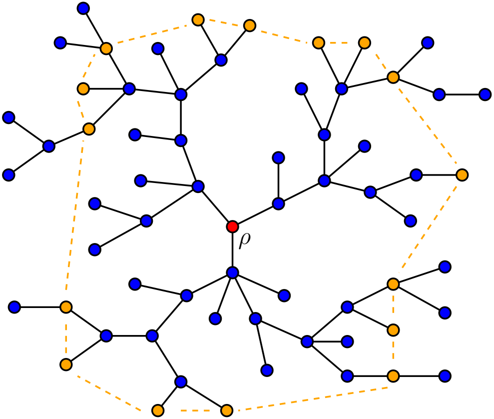
Typically the diameter is $\approx \sqrt{n}$. Let $k \leq \sqrt{n}$. Consider the open ball of radius $k$ centered at $\rho$. (Here $k = 5$.)

Condition on the event that there are vertices outside the ball (conditioning is superfluous if $k = o(\sqrt{n})$).

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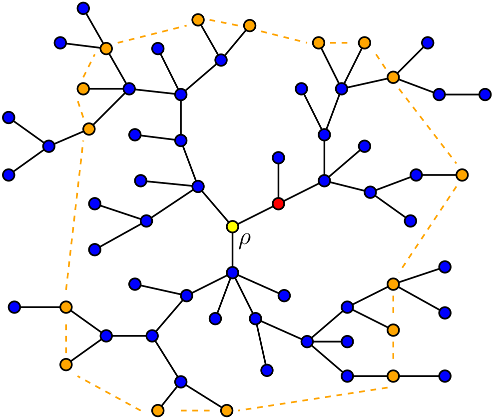


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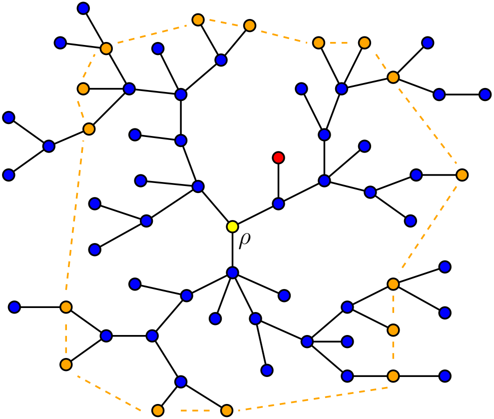


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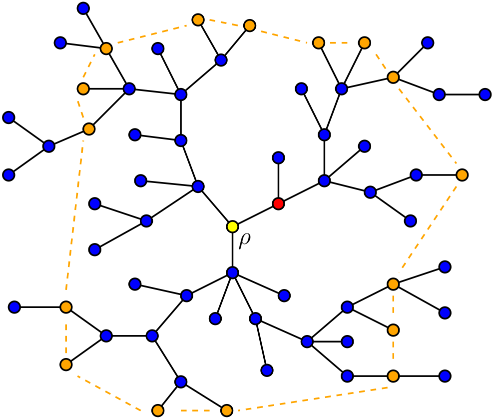


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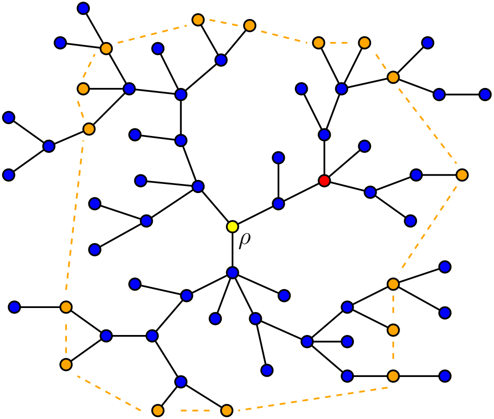


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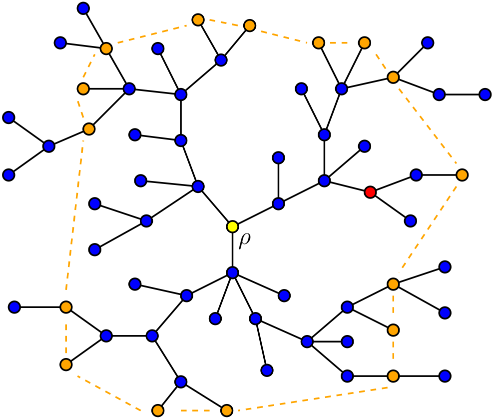


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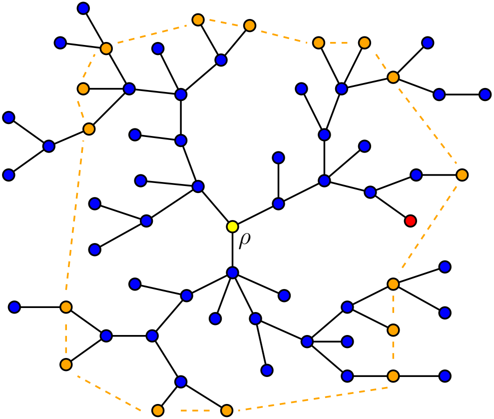


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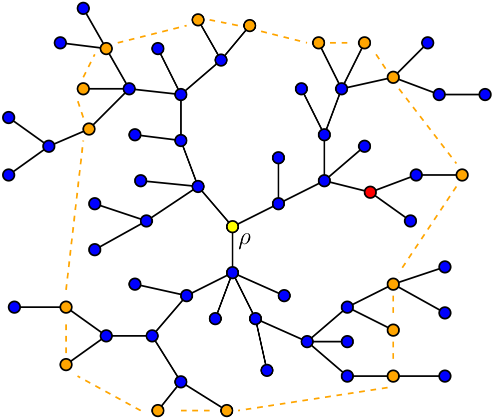


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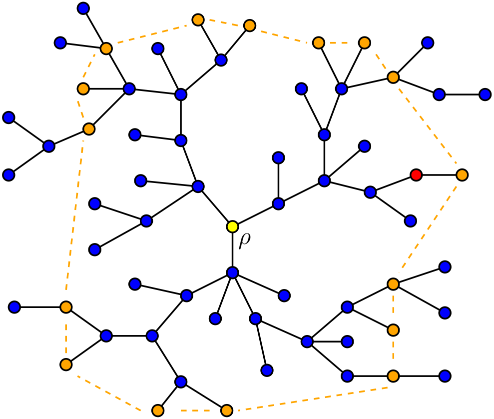


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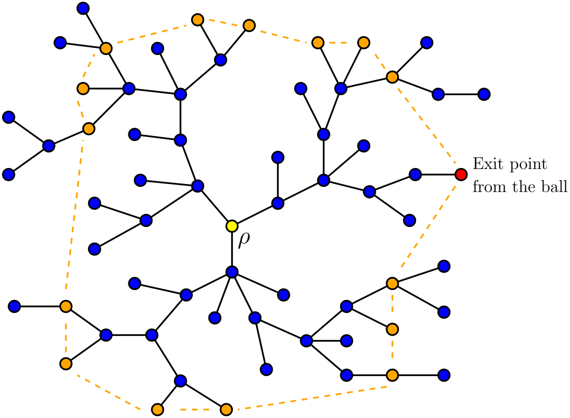


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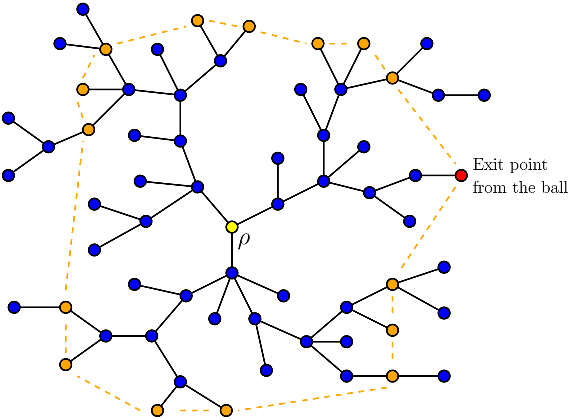


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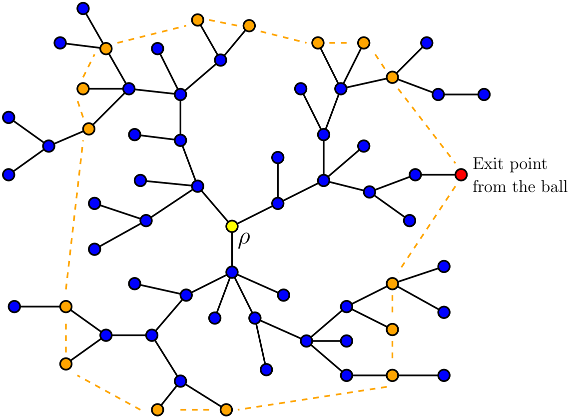
The support of $\mu$ is the set of points at distance $k$ from $\rho$, whose cardinality is of order $k$.

Question (T. Jonsson): As $n, k \to \infty$, with $k \leq \sqrt{n}$ :

– Is harmonic measure "uniformly spread" over vertices at distance $k$?

– Or is most of the harmonic measure supported on a "small subset"?

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**Theorem**

*There is a universal constant $\beta \approx 0.78$ such that, for every $\varepsilon > 0$, with high probability as $n, k \to \infty$ with $k \leq \sqrt{n}$,*

- *there is a set $A$ such that $\#(A) \leq k^{\beta+\varepsilon}$ and $\mu(A) \geq 1 - \varepsilon$,*
- $\max\{\mu(A) : \#(A) \leq k^{\beta-\varepsilon}\} \leq \varepsilon$.

# Remarks

- Since $\beta = 0.78...$, and recalling that the boundary of the ball has size $\approx k$, we see that:

  Harmonic measure is supported on a small subset (size $\approx k^{\beta}$).

  True as $k \to \infty$, uniformly in $n \geq k^2$ (can have $k << \text{diameter}$).

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- A similar "dimension drop" phenomenon has been observed in the case of (supercritical Galton-Watson) trees for the harmonic measure at infinity, cf Lyons, Pemantle, Peres (1995,1996)
- More precise version of the theorem:

$$\mu(\{v : k^{-\beta-\varepsilon} \leq \mu(v) \leq k^{-\beta+\varepsilon}\}) \longrightarrow 1$$

  as $n, k \to \infty$ with $k \leq \sqrt{n}$.

# Universality of $\beta$

Key fact: $\beta$ does not depend on the class of combinatorial trees that is considered.

In fact, the result holds more generally for

Galton-Watson trees conditioned to have *n* edges

$\longrightarrow$ Includes Cayley trees, plane trees = rooted ordered trees, binary plane trees, etc.

Remark. The scaling limit of these conditioned Galton-Watson trees is the CRT (Aldous, 1993).

More generally, one expects the result to hold, with the same $\beta$ for other combinatorial trees whose scaling limit is also given by the CRT, for instance rooted unordered trees (cf Haas, Miermont (2012))

# 2. Galton-Watson trees

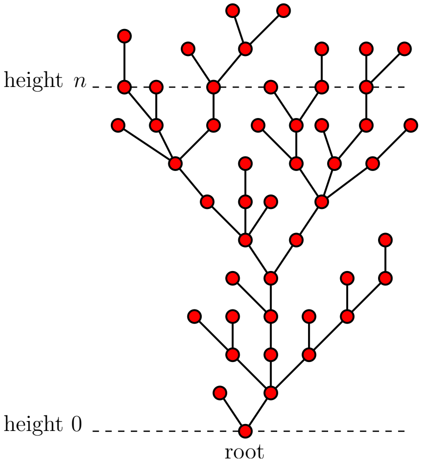
Let $\theta$ be a probability measure on $\{0, 1, \ldots\}$, such that $\theta(1) < 1$ and

- $\displaystyle\sum_{k=0}^{\infty} k\theta(k) = 1$ (critical)
- $\displaystyle\sum_{k=0}^{\infty} k^2\theta(k) < \infty$ (finite variance)

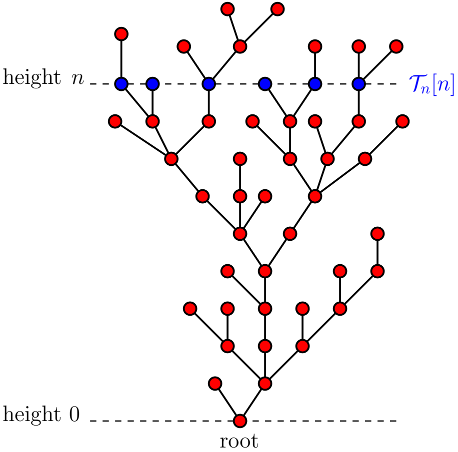
The Galton-Watson tree with offspring distribution $\theta$ (in short GW($\theta$)) is the genealogical tree of a population starting with one ancestor or root, where each individual has $k$ children with probability $\theta(k)$. This tree is finite a.s.

Let $\mathcal{T}_n$ be a GW($\theta$) conditioned to have height at least $n$, and

$$\mathcal{T}_n[n] = \{\text{vertices of } \mathcal{T}_n \text{ at height } n\}.$$




$\mathcal{T}_n$ Galton-Watson tree conditioned to have height at least $n$

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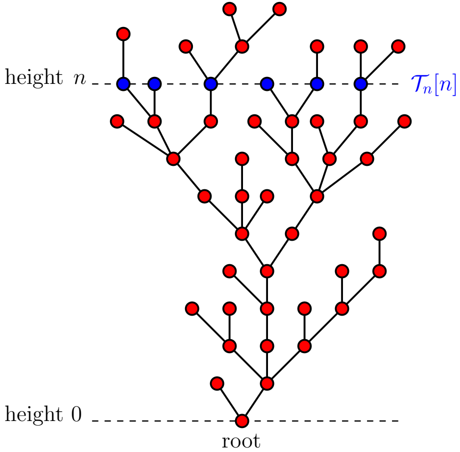


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It is well known that $\#\mathcal{T}_n[n] \approx n$, more precisely $\frac{1}{n}\#\mathcal{T}_n[n] \xrightarrow[n\to\infty]{(d)}$ exp.

$\mu_n$ distribution of hitting point of $\mathcal{T}_n[n]$ by simple random walk on $\mathcal{T}_n$ started from root

$\mu_n$ distribution of hitting point of $\mathcal{T}_n[n]$ (= generation $n$ of a GW tree $\mathcal{T}_n$ conditioned to have height $\geq n$) by simple random walk on $\mathcal{T}_n$

Recall $\#\mathcal{T}_n[n] \approx n$: Naively, one might expect $\mu_n(v) \approx \frac{1}{n}$ for $v \in \mathcal{T}_n[n]$??

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$$\mu_n(\{v \in \mathcal{T}_n[n] : n^{-\beta-\varepsilon} \leq \mu_n(v) \leq n^{-\beta+\varepsilon}\}) \xrightarrow[n\to\infty]{(P)} 1.$$

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Consequences.

- For $\delta > 0$, there exists with probab. $\to 1$ a subset $A \subset \mathcal{T}_n[n]$ s.t.

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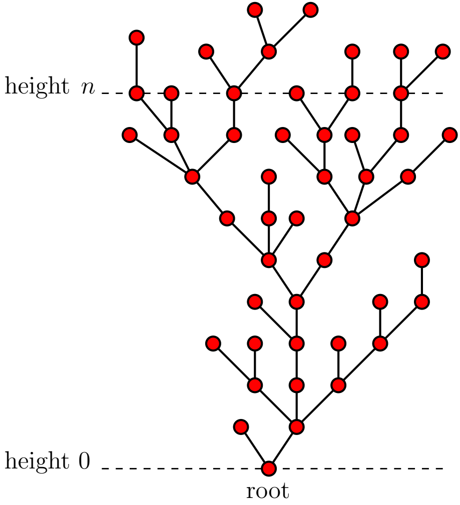
- The theorem for combinatorial trees can be deduced from Theorem (A) (modulo some technical work).

# 3. The continuous setting

Key idea : consider reduced trees.

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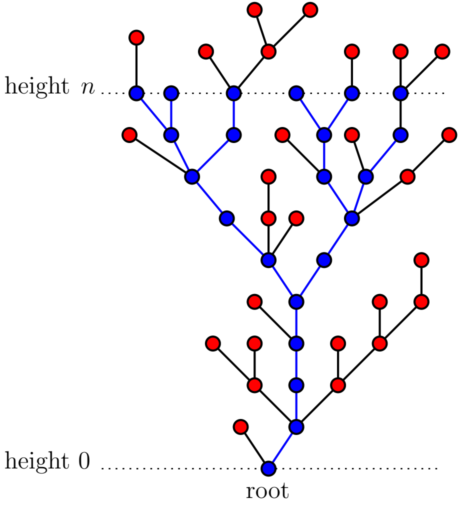


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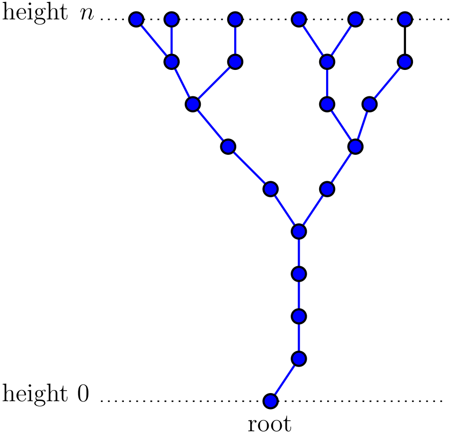


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The hitting distribution of $\mathcal{T}_n[n]$ is the same for SRW on $\mathcal{T}_n^*$ as for SRW on $\mathcal{T}_n$

# A large reduced tree

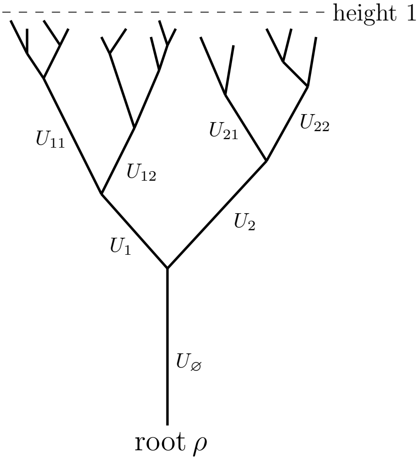

A large Galton-Watson tree and the corresponding reduced tree.

Asymptotics for reduced critical Galton-Watson trees have been studied by Fleischmann and Siegmund-Schultze (1977) following Zubkov.

# Convergence of reduced trees

$d_{\mathrm{gr}}$ graph distance on $\mathcal{T}_n^*$

**Fact**: $(\mathcal{T}_n^*, \frac{1}{n} d_{\mathrm{gr}}) \xrightarrow[n\to\infty]{(d)} (\mathcal{T}_\infty^*, D)$ in the Gromov-Hausdorff sense.




The tree $\mathcal{T}_\infty^*$.

$U_\varnothing$ uniform on $[0, 1]$

$U_1, U_2$ uniform on $[0, 1 - U_\varnothing]$

$U_{11}, U_{12}$ uniform on $[0, 1 - U_\varnothing - U_1]$

and so on.

$D$ is the tree metric on the (completion of the) union of the segments

By definition $\partial\mathcal{T}_\infty^* := \{x \in \mathcal{T}_\infty^* : D(\rho, x) = 1\}$.

# Harmonic measure on $\partial\mathcal{T}^*_\infty$

Let $\Gamma_t$ be Brownian motion on $\mathcal{T}^*_\infty$
$\longrightarrow$ easy to define up to $T := \inf\{t \geq 0 : \Gamma_t \in \partial\mathcal{T}^*_\infty\}$

(at each branching point, Brownian motion chooses with equal probabilities each of the three possible directions)

Let $\mu$ be the law of $\Gamma_T$ (probab. measure on $\partial\mathcal{T}^*_\infty$)

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**Theorem (B)**

*A.s.,* $\mu(\mathrm{d}x)$ *a.e.,*

$$\lim_{r\to 0} \frac{\log \mu(B_D(x,r))}{\log r} = \beta$$

*In particular,* $\dim \mu = \beta$.

**Note**: $\dim(\partial\mathcal{T}^*_\infty) = 1$ a.s. (dimension drop as in Makarov's theorem)

This theorem is a key ingredient of the proof of the discrete results.
(also explains why $\beta$ is universal in the discrete setting)

# 4. The Yule tree

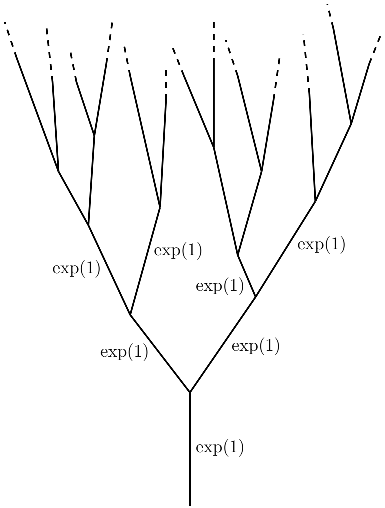
Scale the heights in $\mathcal{T}^*_\infty$ with

$$h(r) = -\log(1-r)$$

Then $\mathcal{T}^*_\infty$ is transformed in the Yule tree $\mathbb{T}$
= genealogical tree of population with

- binary branching
- exp(1) lifetimes

$\Gamma$ (Brownian motion on $\mathcal{T}^*_\infty$) is transformed (up to time change) in
$W$ Brownian motion on $\mathbb{T}$ with drift $\frac{1}{2}$ upwards




The Yule tree $\mathbb{T}$

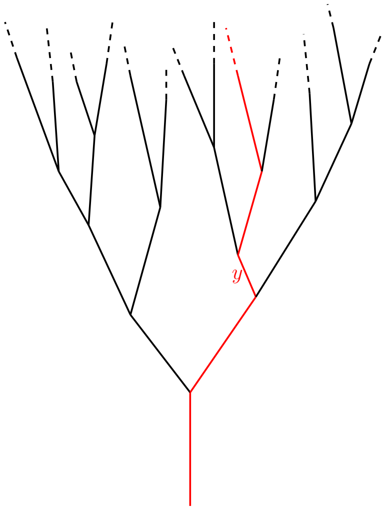
# The boundary of the Yule tree

$\mathbb{T}$ Yule tree
The boundary $\partial\mathbb{T}$ is

$$\partial\mathbb{T} := \{\text{geodesic rays}\}$$

One can define $W_\infty$
the exit ray of
the Brownian motion $W$

(unique ray visited by $W$
at arbitrarily large times)




The Yule tree $\mathbb{T}$
and a particular geodesic ray $y$

# Asymptotics for the law of the exit ray

Recall: $W_\infty$ exit ray of Brownian motion (with drift $\frac{1}{2}$) on the Yule tree $\mathbb{T}$
Set:

$$\nu = \text{law of } W_\infty \text{ (probability measure on } \partial\mathbb{T})$$

For $y \in \partial\mathbb{T}$ and $r > 0$, let

$$\mathcal{B}(y,r) = \{\text{geodesic rays that coincide with } y \text{ up to height } r\}$$

An equivalent form of Theorem (B) is:

**Theorem (C)**

*A.s., $\nu(\mathrm{d}y)$ a.e.,*

$$\lim_{r\to\infty} \frac{1}{r} \log \nu(\mathcal{B}(y,r)) = -\beta.$$

End of the lecture: ideas for the proof of Theorem (C).

# 5. The conductance of the Yule tree

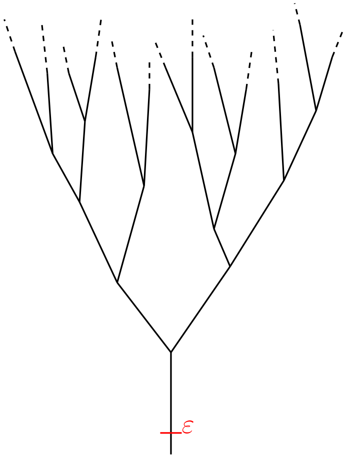
Let $\mathcal{T}$ be a Yule-type (deterministic) tree.

Write $W$ for Brownian motion with drift $\frac{1}{2}$ on $\mathcal{T}$.

The conductance of $\mathcal{T}$ is

$$\mathcal{C}(\mathcal{T}) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} P_\varepsilon(W \text{ never hits the root})$$

where under $P_\varepsilon$, the Brownian motion $W$ starts at distance $\varepsilon$ from the root.




A Yule-type tree $\mathcal{T}$

# The law of the conductance of the Yule tree

Recall: $\mathbb{T}$ is the Yule tree.

Let $\gamma(\mathrm{d}s)$ be the law of the conductance $\mathcal{C} := \mathcal{C}(\mathbb{T})$ of the Yule tree.

## Proposition

$\gamma(\mathrm{d}s)$ *is characterized by the recursive equation in distribution*

$$\mathcal{C} \overset{(\mathrm{d})}{=} \left( U + \frac{1-U}{\mathcal{C}_1 + \mathcal{C}_2} \right)^{-1}$$

*where*

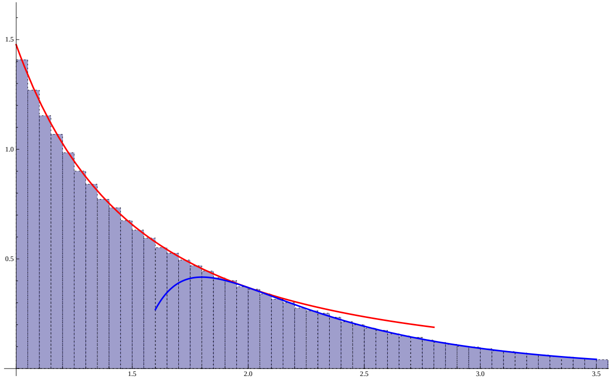
- $U$ *is uniform over* $[0,1]$
- $\mathcal{C}_1, \mathcal{C}_2$ *are copies of* $\mathcal{C}$
- $U, \mathcal{C}_1, \mathcal{C}_2$ *are independent*

(recursive equation easy from electrical networks theory)

Let $\varphi$ be the Laplace transform of $\gamma$

$$\varphi(\lambda) = \int \gamma(\mathrm{d}s)\, e^{-\lambda s/2} = E[e^{-\lambda \mathcal{C}/2}]$$

# The density of the law of the conductance of the Yule tree




A histogram of the distribution $\gamma(ds)$ over $[1, \infty)$

The red curve is the density over $[1, 2]$ (known up to a parameter)
The blue curve is the density over $[2, 3]$ (known up to a parameter)

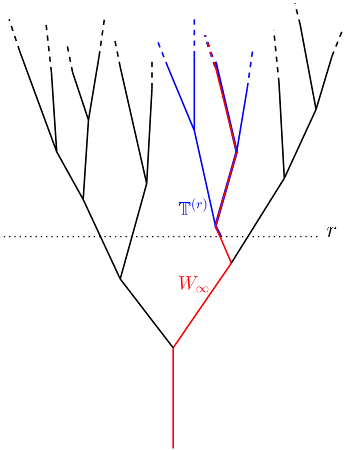
# 6. The subtree selected by Brownian motion

$\mathbb{T}$ Yule tree
$W_\infty$ exit ray of Brownian motion

For every $r > 0$, define $\mathbb{T}^{(r)}$ subtree of $\mathbb{T}$ above $r$ that "contains" $W_\infty$

$\mathbb{T}^{(r)}$ is a (random) Yule-type tree
BUT its law is not the same as the law of the Yule tree $\mathbb{T}$

(harmonic measure induces a bias: favors subtrees with a large conductance!)




The Yule tree $\mathbb{T}$
In red, the exit ray $W_\infty$
In blue, the subtree $\mathbb{T}^{(r)}$

# The law of the subtree $\mathbb{T}^{(r)}$

Let $\Theta(\mathrm{d}\mathcal{T})$ be the law of the Yule tree $\mathbb{T}$. Recall :

- $\mathcal{C}(\mathcal{T})$ = conductance of $\mathcal{T}$
- $\varphi(\lambda) = \int \exp(-\lambda\,\mathcal{C}(\mathcal{T})/2)\,\Theta(\mathrm{d}\mathcal{T})$

## Proposition

*The law of the subtree $\mathbb{T}^{(r)}$ above level $r$ selected by Brownian motion is*

$$\Phi_r(\mathcal{C}(\mathcal{T}))\,\Theta(\mathrm{d}\mathcal{T})$$

*where, for any $c > 0$,*

$$\Phi_r(c) = E_{(c)}\Big[\exp - \int_0^r \mathrm{d}s\,(1 - 2\varphi(X_s))\Big]$$

*here $X$ stands for the solution of the SDE*

$$\mathrm{d}X_s = 2\sqrt{X_s}\,\mathrm{d}B_s + (2 - X_s)\,\mathrm{d}s$$

*that starts under $P_{(c)}$ with an $\exp(\frac{c}{2})$ distribution.*

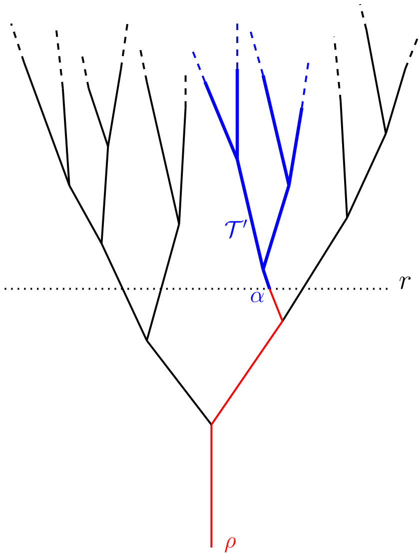
# Idea of proof

In order to escape in the blue subtree $\mathcal{T}'$, Brownian motion must accumulate at its root $\alpha$ a local time exp. with parameter $\mathcal{C}(\mathcal{T}')$

(follows from the definition of the conductance!)

Then one must say that Brownian motion did not escape before in one of the subtrees branching off the red segment joining $\rho$ to $\alpha$

- These subtrees form a Poisson process
- Needs the law of local times along the red segment (Ray-Knight theorem)



# Asymptotics for the law of the subtree selected by BM

Recall

- the law of the subtree above level $r$ selected by Brownian motion has density $\Phi_r(\mathcal{C}(\mathcal{T}))$ w.r.t. the law $\Theta(\mathrm{d}\mathcal{T})$ of the Yule tree
- $\gamma(\mathrm{d}s)$ is the law of $\mathcal{C}(\mathcal{T})$ under $\Theta(\mathrm{d}\mathcal{T})$

## Proposition

*We have*

$$\Phi_r(c) \underset{r\to\infty}{\longrightarrow} \Phi_\infty(c)$$

*where*

$$\Phi_\infty(c) = \frac{\int \gamma(\mathrm{d}s)\frac{cs}{c+s-1}}{\iint \gamma(\mathrm{d}s)\gamma(\mathrm{d}t)\,\frac{st}{s+t-1}}$$

**Proof.** Stochastic analysis, Girsanov theorem, etc.

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**Proof.** Stochastic analysis, Girsanov theorem, etc.

**Application.** $\Phi_\infty(\mathcal{C}(\mathcal{T}))\,\Theta(d\mathcal{T})$ will be an invariant measure for the process $r \longrightarrow \mathbb{T}^{(r)}$

$\longrightarrow$ tools from ergodic theory! cf Lyons, Pemantle, Peres (1995,1996)

# 7. Ergodic theory

$\Omega = \{\text{Yule-type trees}\}$

$\Omega^* = \{(\mathcal{T}, y) : \mathcal{T} \in \Omega, y \text{ ray of } \mathcal{T}\}$

Shifts on $\Omega^*$:

$\tau_r(\mathcal{T}, y) = (\mathcal{T}', y')$, where

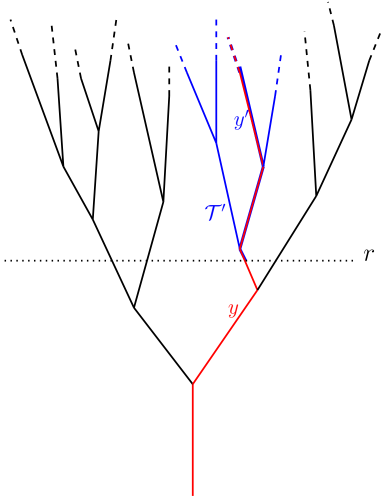
- $\mathcal{T}'$ is the subtree above level $r$ "containing" $y$
- $y'$ is the ray in $\mathcal{T}'$ corresponding to $y$

$\Theta^*(\mathrm{d}\mathcal{T}\,\mathrm{d}y)$ = law of $(\mathbb{T}, W_\infty)$

$\Theta^*$ is NOT invariant under the shifts, BUT

$\Lambda^*(\mathrm{d}\mathcal{T}\,\mathrm{d}y) = \Phi_\infty(\mathcal{C}(\mathcal{T}))\Theta^*(\mathrm{d}\mathcal{T}\,\mathrm{d}y)$

is invariant (and ergodic)




A pair $(\mathcal{T}, y) \in \Omega^*$ and the shift $(\mathcal{T}', y') = \tau_r(\mathcal{T}, y)$

$\mathcal{T}'$ is the blue subtree

# Applying Birkhoff's ergodic theorem

For $(\mathcal{T}, y) \in \Omega^*$ (= pairs consisting of a tree + a geodesic ray), set

$$F_r(\mathcal{T}, y) = -\log \nu_{(\mathcal{T})}(\mathcal{B}_{(\mathcal{T})}(y, r))$$

where

- $\nu_{(\mathcal{T})} =$ harmonic measure on $\partial\mathcal{T}$
  (law of exit ray for Brownian motion on $\mathcal{T}$)
- $\mathcal{B}_{(\mathcal{T})}(y, r) =$ rays of $\mathcal{T}$ that coincide with $y$ up to height $r$

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Then, for every $r, s \geq 0$,

$$F_{r+s} = F_r + F_s \circ \tau_r$$

(conditionally on the event that Brownian motion escapes in a subtree $\mathcal{T}'$ above height $r$, the law of the exit ray is given by the harmonic measure of $\mathcal{T}'$)

By Birkhoff's theorem,

$$\frac{1}{r} F_r \underset{r\to\infty}{\longrightarrow} \beta := \Lambda^*(F_1)$$

$\Lambda^*$ a.s. hence also $\Theta^*$ a.s. (recall $\Theta^*$ has a density w.r.t. $\Lambda^*$)

# End of the proof

We have obtained, $\Theta^*(\mathrm{d}\mathcal{T}\,\mathrm{d}y)$ a.s.

$$-\frac{1}{r}\log \nu_{(\mathcal{T})}(\mathcal{B}_{(\mathcal{T})}(y,r)) \underset{r\to\infty}{\longrightarrow} \beta$$

But $\Theta^*(\mathrm{d}\mathcal{T}\,\mathrm{d}y)$ is the law of $(\mathbb{T}, W_\infty)$, so this is equivalent to Theorem (C):

$$\frac{1}{r}\log \nu(\mathcal{B}(W_\infty, r)) \xrightarrow[r\to\infty]{\text{a.s.}} -\beta$$

# End of the proof

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Computing $\beta$ : From formula for $\Phi_\infty$ one gets

$$\beta = 2\, \frac{\iiint \gamma(dr)\gamma(ds)\gamma(dt)\, \frac{rs}{r+s+t-1} \log \frac{r+t}{r}}{\iint \gamma(dt)\gamma(ds) \frac{rs}{r+s-1}}$$

where $\gamma(ds)$ is the law of the conductance of the Yule tree
(not so easy with this formula to see that $\beta < 1$ !)

# 8. Further results (Lin 2014)

## Behavior of the harmonic measure near a typical point of $\mathcal{T}_n[n]$

Let $\lambda_n$ be the uniform probability measure on $\mathcal{T}_n[n]$ (generation $n$ of the tree). There exists an exponent $\gamma > 1$ such that:

$$\lambda_n(\{v \in \mathcal{T}_n[n] : n^{-\gamma-\varepsilon} \leq \mu_n(v) \leq n^{-\gamma+\varepsilon}\}) \xrightarrow[n\to\infty]{(P)} 1.$$

Furthermore, $\gamma = E[\widehat{C}] - 1$ where $\widehat{C}$ is the conductance of the continuous reduced tree $\mathcal{T}^*_\infty$ "size-biased" at generation 1.

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## The stable case

For critical Galton-Watson trees whose offspring distribution is in the domain of attraction of a stable law with index $\alpha \in (1, 2)$:

- similar result for the harmonic measure with an exponent $\beta_\alpha$;
- $\beta_\alpha < \frac{1}{\alpha-1}$, which is the dimension of the boundary;
- $\beta_\alpha$ remains bounded when $\alpha \downarrow 1$.