Conformal invariance of the Green's function for loop-erased random walk

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OUTLINE OF TALK

- Statement of main theorem (w. C. Beneš and F.Johansson Viklund) on planar loop-erased random walk (LERW)
- Restatement in terms of scaling limit
- History of previous results
- Natural parametrization and conjecture on scaling limit of LERW
- Ideas behind proof
  - Loop measures (random walk and Brownian motion)
  - Negative weights ("zippers" of Kenyon; "spinors")
  - Combinatorial identity
  - Estimating the quantities in the limit

Setup for problem

- A — finite simply connected subset of $\mathbb{Z}^2 = \mathbb{Z} + i\,\mathbb{Z}$ containing the origin. $\partial A = \{z : \mathrm{dist}(z, A) = 1\}$.
- Associated to A is a simply connected domain D_A which is the interior of the union of the closed squares of side one containing $\zeta \in A$.
- a, b distinct elements of the "edge boundary" of A, $a = (w_a, z_a), b = (z_b, w_b)$ with $w_a, w_b \in \partial A$, $z_a, z_b \in A$. We also write a, b for the midpoints of these edges which are on ∂D_A.
- $f_A : D \to \mathbb{D}$ the conformal transformation with $f_A(0) = 0, f_A(a) = 1$. We define θ by $f(b) = e^{2i\theta}$.
- $r_A = 1/|f_A'(0)|$ is the conformal radius of D_A (with respect to 0).

- We write $p(x, y) = 1/4$ if $|x - y| = 1$ for the usual random walk edge weight.
- $\omega = [\omega_0, \omega_1, \ldots, \omega_n]$ for a nearest neighbor path

$$p(\omega) = \prod_{j=1}^{n} p(\omega_{j-1}, \omega_j) = (1/4)^{|\omega|}.$$

- If a, b are boundary edges in A, we let $\mathcal{K}_A(a, b)$ denote the set of paths $\omega : a \to b$ in A.
- $$H_A(a, b) = \sum_{\omega \in \mathcal{K}_A(a,b)} p(\omega).$$

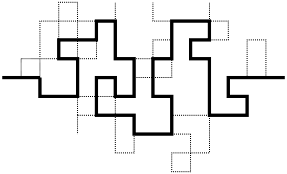
  In this case it is just the probability that a random walk starting with edge a exits A at edge b. [(boundary) Poisson kernel, partition function]

- A path is a self-avoiding walk (SAW) if $\omega_j \neq \omega_k$ for $j < k$. We will use

$$\eta = [\eta_0, \ldots, \eta_k],$$

for SAWs.

- Let $\mathcal{W}_A(a, b)$ denote the set of paths in $\mathcal{K}_A(a, b)$ that are self-avoiding walks.
- There is a deterministic operation that assigns to each $\omega \in \mathcal{K}_A(a, b)$ a self-avoiding subpath $LE(\omega) \in \mathcal{W}_A(a, b)$ by (chronological) loop erasure.



- We define

$$\hat{p}(\eta) = \hat{p}_A(\eta; a, b) = \sum_{\omega \in \mathcal{K}_A(a,b), LE(\omega)=\eta} p(\omega),$$

  Then

$$\sum_{\eta \in \mathcal{W}_A(a,b)} \hat{p}(\eta) = H_A(a, b).$$

- Let $\mathcal{W}^+$ ($\mathcal{W}^-$) denote the set $\eta \in \mathcal{W}_A(a, b)$ that use the directed edge $\vec{01}$ (resp., $\vec{10}$). Let $\mathcal{W}^* = \mathcal{W}^+ \cup \mathcal{W}^-$.
- The probability that the loop-erased walk from a to b in A goes through the undirected edge $(0, 1)$ is

$$H_A(a, b)^{-1} \sum_{\eta \in \mathcal{W}^*} \hat{p}(\eta).$$

Theorem (with C. Beneš and F. Johansson Viklund)

There exist $c < \infty$ and $u > 0$ such that if A is a simply connected subset of $\mathbb{Z} \times i\mathbb{Z}$ containing the origin, and a, b are points on the edge boundary, then the probability that the LERW from a to b uses the directed edge $\vec{01}$ is

$$c\, r_A^{-3/4}\, \left[\sin^3\theta + O(r_A^{-u})\right].$$

- The difference between the probability of using $\vec{01}$ and using $\vec{10}$ is easier (and known). The difference is $O(r_A^{-1})$.
- For any finite SAW $\tilde{\eta}$ containing the origin there is a similar expression for the probability that the LERW contains $\tilde{\eta}$.

- Suppose D is a Jordan domain containing the origin and a, b are distinct points in ∂D.
- Let $f : D \to \mathbb{D}$ with $f(0) = 0, f(a) = 1, f(b) = e^{2i\theta}$ and $r_D = |f'(0)|^{-1}$.
- Suppose we put in D a grid with lattice spacing $1/n$ and let a_n, b_n be boundary points near a, b.
- Then as $n \to \infty$, the probability that the loop-erased walk goes though the unordered edge $[0, 1/n]$ is asymptotic to $c\, n^{-3/4}\, r_A^{-3/4}\, \sin^3\theta$.
- The function $G_D(z) = r_D(z)^{-3/4}\, \sin^3 \arg_D(z; a, b)$ is the Green's function for the Schramm-Loewner evolution (SLE) with parameter 2. If γ is an SLE_2 path from a to b in D then the probability that it gets within distance ϵ of 0 is asymptotic to

$$\hat{c}\, \epsilon^{3/4}\, r_D^{-3/4}\, \sin^3\theta.$$

PREVIOUS RESULTS FOR 2-D LERW

- Majumdar and Duplantier (independently) gave nonrigorous predictions of the exponent $\frac{3}{4}$ (often phrased as the fractal dimension $d = 2 - \frac{3}{4} = \frac{5}{4}$).
- Kenyon (2000) used relation between LERW and dimers and uniform spanning trees as well as another idea (zippers) to show that the probability that of going through the edge $0, 1$ in a square is logarithmically asymptotic to $n^{-3/4}$. He also gave a partial result giving the angular dependence.
- Schramm (2000) showed that if LERW has a conformally invariant limit, then in some sense the limit is SLE_2.
- L, Schramm, Werner (2003) showed that the scaling limit of LERW in the capacity parametrization is SLE_2. The exponent $3/4$ (as an intersection exponent) was proved rigorously for SLE.

- Masson (2008) used the *SLE* results to give an alternative proof that the probability for LERW was logarithmically asymptotic to $n^{-3/4}$. His result was more universal than that of Kenyon and LSW.
- Yadin and Yehudayoff (2011) gave a different unversality result for planar graphs.
- L (2014) proved a version of the combinatorial identity and used it to give up-to-constants estimates for LERW in a square. (Although not directly related, recent work of Kenyon and Wilson was useful in helping to understand the key ideas in Kenyon's 2000 paper.)

NATURAL PARAMETRIZATION CONJECTURE

- Suppose D is a simply connected domains with (say) analytic boundary and distinct boundary points a, b.
- Approximate D by a lattice of spacing $1/n$. Consider LERW from a to b as a probability measure.
- If Y_n denotes the number of steps, then $Y_n \asymp n^{5/4}$.
- CONJECTURE: In fact, $Y_n/n^{5/4}$ has a limit distribution (the distribution depends on D, a, b.)
- For each scaled SAW η at level n, reparametrize time, that is if $\eta = [\eta_0, \ldots, \eta_k]$, consider

$$\eta(t) = \frac{\eta_{tn}}{n}. \quad 0 \leq t \leq \frac{k}{n^{5/4}}.$$

- CONJECTURE: this converges to SLE_2 in the natural parameterization.

NATURAL PARAMETRIZATION

- The Green's function for SLE_2 is given by

$$\begin{aligned} G_D(z;a,b) &= \lim_{\epsilon \downarrow 0} \epsilon^{-3/4}\, \mathbb{P}\{\mathrm{dist}(\gamma, z) < \epsilon\} \\ &= \hat{c}\, r_D(z)^{-3/4} \sin^3 \theta_D(z;a,b) \end{aligned}$$

  (Rohde-Schramm,...,L-Rezaei)
- Using this as motivation, L-Sheffield constructed directly for SLE_2 the candidate for the scaled limit of the number of steps of the walk. It was called natural parametrization or natural length.

- (L-Rezaei) The natural parametrization is given by the Minkowski content

$$\mathrm{Cont}_{5/4}(\gamma[0,t]) = \lim_{\epsilon \downarrow 0} \epsilon^{-5/4} \, \mathrm{area}\{z : \mathrm{dist}(z, \gamma[0,t]) < \epsilon\}.$$

- Alberts, Kozdron, Masson (2013) gave a program to establish the convergence of SLE_2 in natural parametrization.
- The key step in making this or other such programs work is to establish the limit in our result.
- We are optimistic (but have not yet proved) that our result can establish the convergence result.

Rooted Loop measure

- Although this can be done in more generality, we only consider the random walk measure p and a second measure of form

$$q(x,y) = Q(x,y)\, p(x,y), \qquad Q(x,y) = Q(y,x) = \pm 1,$$

$$q(\omega) = \prod_{j=1}^{n} q(\omega_{j-1}, \omega_j) = \pm p(\omega)$$

- A rooted loop (in $\mathbb{Z}^2$) is a nearest neighbor path $\omega = [\omega_0, \ldots, \omega_{2n}]$ with $\omega_0 = \omega_{2n}$.
- Let $\mathcal{O}(A)$ denote the set of (rooted) loops that stay in A.
- The rooted loop measures are given by

$$m(\omega) = \frac{p(\omega)}{|\omega|}, \qquad m^q(\omega) = \frac{q(\omega)}{|\omega|} = \pm m(\omega).$$

for $\omega \in \mathcal{O}(A)$ with $|\omega| \geq 1$.

Unrooted loops

- An unrooted loop $\tilde{\omega} \in \tilde{\mathcal{O}}$ is an equivalence class of rooted loops generated by the equivalence

$$[\omega_0, \ldots, \omega_n] \sim [\omega_1, \ldots, \omega_n, \omega_1].$$

- In other words, an unrooted loop is a loop that has forgotten its starting point (but not its orientation).
- We write $\omega \sim \tilde{\omega}$ if ω is in the equivalence class for $\tilde{\omega}$.
- The unrooted loop measure is defined by

$$\tilde{m}(\tilde{\omega}) = \sum_{\omega \sim \tilde{\omega}} m(\omega), \quad \tilde{m}^q(\tilde{\omega}) = \sum_{\omega \sim \tilde{\omega}} m^q(\omega).$$

Relationship between Loop Measure and LERW

(See, e.g., L-Limic, Random Walk, A Modern Introduction)

- If $V \subset A$, let If $V \subset A$.

$$F_V(A) = \exp\left\{ \sum_{\omega \in \mathcal{O}(A), \omega \cap V \neq \emptyset} m(\omega) \right\}.$$

$$F(A) = F_A(A) = \exp\left\{ \sum_{\omega \in \mathcal{O}(A)} m(\omega) \right\} = \exp\left\{ \sum_{\tilde{\omega} \in \tilde{\mathcal{O}}(A)} \tilde{m}(\tilde{\omega}) \right\}.$$

- Define $F_V^q(Z), F^q(A)$ similarly with m^q.

- Recall that if A finite, simply connected subset of $\mathbb{Z}^2$ with boundary edges $a = (w_a, z_a), b = (z_b, w_b)$.

$$H_A(a,b) = \sum_{\omega \in \mathcal{K}(a,b)} p(\omega),$$

$$\hat{p}(\eta) = \sum_{\omega \in \mathcal{K}(a,b), LE(\omega)=\eta} p(\omega).$$

- Define $H^q_A(a,b), \hat{q}(\eta)$ similarly.
- Straightforward analysis of loop-erasing shows that

$$\hat{p}(\eta) = p(\eta)\, F_\eta(A), \quad \hat{q}(\eta) = q(\eta)\, F^q_\eta(A).$$

Zipper

- $A \subset \mathbb{Z} \times i\mathbb{Z}$, simply connected containing origin.
- $w_0 = \frac{1}{2} - \frac{i}{2}$.
- Draw a vertical line from w_0 downward to ∂A.
- Set $q(z,w) = -1/4$ if $\{z,w\}$ crosses the zipper. Otherwise $q(z,w) = 1/4$.
- Closely related to spinors.

Observable

- If ω is a path, let $Y^+(\omega), Y^-(\omega)$ denote the number of traverses of the ordered edge $\vec{01}$ (resp., $\vec{10}$) and $Y = Y^+ - Y^-$ the number of signed traverses.

Identity for the "expectation value" of the observable

- Let

$$\Lambda = \langle Y \rangle_{q,A,a,b} = \sum_{\omega \in \mathcal{K}(a,b)} q(\omega)\, Y(\omega).$$

- We give two different expressions for Λ and then equate them.
- First expression:

$$\Lambda = \exp\{-2m(\mathcal{J})\} \sum_{\mathbf{e} \subset LE(\omega)} p(\omega)$$

  where the sum is over all ω whose loop erasure uses the undirected edge $\mathbf{e} = \{0, 1\}$ and $\mathcal{J} = \mathcal{J}_A$ is the set of loops in A with odd winding number about w_0.

  - A key topological fact that is used is that if η is a SAW going through $0, 1$ and l is a loop with odd winding number about w_0 then $l \cap \eta \neq \emptyset$.

- Let $\tilde{A} = A \setminus \{0,1\}$,

$$\Delta^q(A;a,b) = \left| H^q_{\partial \tilde{A}}(0,a)\, H^q_{\partial \tilde{A}}(1,b) - H^q_{\partial \tilde{A}}(1,a)\, H^q_{\partial \tilde{A}}(0,b) \right|.$$

- Using a determinant formula (Fomin's identity) applied to the signed measure q, we get the second expression

$$\Lambda = \frac{1}{4}\, F^q_{0,1}(A)\, \Delta^q(A;a,b).$$

  - A cancellation of signs in Fomin's identity when applied to q can be considered the primary idea from Kenyon's work that we use.

- Equating the two expressions gives

$$\sum_{\mathbf{e} \subset LE(\omega)} p(\omega) = \exp\{2m(\mathcal{J}_A)\}\, \frac{1}{4}\, F^q_{0,1}(A)\, \Delta^q(A;a,b).$$

- Our expression does not uses anything from dimers, spanning trees, or SLE.
- We only need to do estimates for simple random walk (including random walk loop measure.)
- Need asymptotics as $r_A \to \infty$. Recall that we want good estimates

$$\exp\{2m(\mathcal{J}_A)\} \frac{1}{4} F^q_{0,1}(A) \, \Delta^q(A; a, b) = \phi(A; a, b) \, [1 + O(r_A^{-u})].$$

- The sharp estimates are needed to solve the natural parametrization conjecture (at least in the programs set out).
- Not difficult to show that $\exists c > 0$ with

$$F^q_{0,1}(A) = c + O(r_A^{-u}).$$

The difficult terms are $\exp\{2m(\mathcal{J}_A)\}$ and $\Delta^q(A; a, b)$.

Theorem

There exists $c_1 \in (-\infty, \infty)$ *and* $u > 0$ *such that*

$$m(\mathcal{J}_A) = \frac{\log r_A}{8} + c_1 + O(r_A^{-u}).$$

Theorem

There exists $c_2 > 0$ *and* $u > 0$ *such that*

$$\frac{\Delta^q(A; a, b)}{\partial H_A(a, b)} = c_2 \, r_A^{-1} \, \left[\sin^3 \theta_A(0; a, b) + O(r_A^{-u})\right].$$

- We do not find the optimal u.
- We probably could determine the constant c_2 but we do not know how to determine c_1.
- The second theorem is proved by analyzing $H^q(0, a), \ldots, H^q(1, b)$ separately.

Rooted Brownian loop measure

- A (rooted) loop γ in $\mathbb{C}$ of time duration t_γ is a continuous function $\gamma : [0, t_\gamma] \to \mathbb{C}$ with $\gamma(0) = \gamma(t_\gamma)$.
- We can write such a loop as $(z, t_\gamma, \tilde{\gamma})$ where $z \in \mathbb{C}$, $t_\gamma > 0$ and $\tilde{\gamma}(0) = \tilde{\gamma}(1) = 0$.
- We get γ from $(z, t_\gamma, \tilde{\gamma})$ by translation and Brownian scaling:

$$\gamma(t) = z + \sqrt{t_\gamma}\, \tilde{\gamma}(t/t_\gamma), \quad 0 \le t \le t_\gamma.$$

- The rooted (Brownian) loop measure is given by

$$(\text{Lebesgue}) \times \left(\frac{dt}{2\pi t^2}\right) \times (\text{Brownian bridge}).$$

- Think of $1/(2\pi t^2)$ as $p_t(0,0)/t$.

(Unrooted) Brownian loop measure (L. - Werner)

- An unrooted root is an equivalence class of rooted loops similarly to the discrete case.
- The (unrooted) Brownian loop measure μ is the measure induced by the rooted loop measure.
- If $D \subset \mathbb{C}$, then μ_D is μ restricted to loops that stay in D.
- The collection of measures $\{\mu_D\}$ satisfy the restriction property: if $D' \subset D$, then $\mu_{D'}$ is μ_D restricted to loops in D';
- For bounded D, μ_D is an infinite measure, but loops of diameter $> \epsilon$ have finite measure.

- The unrooted loop measure is conformally invariant: if $f : D \to f(D)$ is a conformal transformation, then $f \circ \mu_D = \mu_{f(D)}$. (This does not require D to be simple connected.)
- The rooted measure is not conformally invariant.
- The loop measure arises in analysis of SLE and describes how the measure changes when the domain is perturbed.
- (Sheffield-Werner) Brownian loop soups can be used to construct conformal loop ensembles (CLE).

Theorem (L- Trujillo Ferreras, TAMS, 2007)

The Brownian loop measure is the scaling limit of the random walk loop measure.

- More precisely, consider the random walk loop measure where the walks are scaled to the lattice $n^{-1} Z^2$. (The measure is scaled.)
- The limit measure, considered as a measure on macroscopic loops is the Brownian loop measure.
- The loops that "collapse to a point" are thrown away —this is one of many examples of subtracting infinity to get a limit.
- Precise statement uses a coupling and the KMT/Hungarian/dyadic coupling of random walk and Brownian motion.

- The Brownian loop measure of loops in the disk of radius e^{n+1} that are not contained in the disk of radius e^n with odd winding number about the origin is a constant independent of n by conformal invariance. This constant is $1/8$.
- Loops with odd winding number are macroscopic and hence random walk and Brownian loop measures are very close. Can use L. - Trujillo Ferreras coupling to give very good error bounds.
- One has to handle issues about when a random walk loop stays in a domain and a Brownian loop does not (or vice versa). Beurling estimates are used for this.
- One also has to consider then the random walk has odd winding number but the Brownian motion has even winding (or vice versa). This only happens when loops get near the origin in which case the probability of having odd winding number is very close to $1/2$.

- Eventually get that measure of random walk loops in the disk or radius e^{n+1} that are not contained in the disk of radius e^n with odd winding number about the origin equals

$$\frac{1}{8} + O(n^{-2}).$$

  We need $n^{-(1+u)}$ for our result.
- Similarly if D is a domain containing the origin of conformal radius r, then we consider loops with odd winding number about the origin that are contained in D but not in the disk of radius $r/5$.
  - For Brownian loops this can be given in terms of the conformal radius by conformal invariance,
  - For random walk loops use the fact that these are macroscopic loops.

Computing a signed Poisson kernel

- Problem: Given A with weights q given by a vertical "zipper" compute
$$H^q_{\partial \tilde{A}}(0, a), \quad \tilde{A} = A \setminus \{0, 1\}.$$
- Can also be written as
$$\mathbb{E}\left[Q\, S_T\, 1_E\right],$$
where
  - S is a simple random walk starting at the origin,
  - T is the first visit to $\mathbb{Z}^2 \setminus \tilde{A}$ after time 0,
  - E is the event that $S_T = a$.
  - $Q = \pm 1$ depending on the parity of the number of times that the walk crosses the zipper.
- Good guess can be given by a continuous problem for Brownian motion that can be solved explicitly.

- On the microscopic level away from the boundary, we use the detail of the square lattice. (Random walk problem discussed on next slide.)
- After "getting away from the origin" we can use strong (KMT/Hungarian/dyadic) approximation of random walk by Brownian motion.

Random Walk Problem

- Let $A_n = \{z \in \mathbb{Z}^2 : |z| < n\} \setminus [0, \infty)$ be the slit square .
- S be a simple random walk starting at the origin and $T = \min\{j > 0 : S_j \not\in A_n\}$.
- It is known that $\mathbb{P}\{S_T \not\in \mathbb{R}\} \asymp n^{-1/2}$.
- Need

$$\mathbb{P}\{S_T = x + iy\} = c\, n^{-3/2}\, [h(x + iy) + O(n^{-u})],$$

  where $h(x + iy)$ is the prediction given by the (boundary) Poisson kernel for Brownian motion in a slit square.
- We showed a u existed. Open problem: find best u. For a non-slit square, one gets

$$\mathbb{P}\{S_T = x + iy\} = n^{-1}\, [\hat{h}(x + iy) + O(n^{-2})],$$

THANK YOU!