Crossed products from minimal dynamical systems

Huaxin Lin

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if and only if $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$.

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$$\sigma\circ\alpha^{m(x)}(x) = \beta\circ\sigma(x). \tag{e 0.3}$$

where n and m has at most one discontinuites. The fact: $C(X)\rtimes_{\beta}\mathbb{Z}$ is

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(Comm. Math. Phys, **275** *(2005), 425–471)*

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Lemma

The C^*-algebra A_x is locally AH. Moreover, A_x is isomorphic to a unital simple AH-algebra with slow dimension growth. (L–2014)

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Let F_1 and F_2 be two finite dimensional C^*-algebras, and let $\phi_0, \phi_1 : F_1 \to F_2$ be two unital homomorphisms. Denote $A = A(F_1, F_2, \phi_0, \phi_1)$, where

$$A = \{(f, g) : C([0,1], F_2) \oplus F_1 : \phi_0(g) = f(0) \text{ and } \phi_1(g) = f(1)\}$$

These C^*-algebras are called Elliott-Thomsen building blocks. It is the pull-back:

$$\begin{array}{ccc} A & \longrightarrow & C([0,1], F_2) \\ \downarrow_{\pi_e} & & \downarrow_{(\pi_0, \pi_1)} \\ F_1 & \longrightarrow_{(\phi_0, \phi_1)} & F_2 \oplus F_2 \end{array}$$

also called one-dimensional NCCW. The set of all C^*-algebras with the above form is denoted by $\mathcal{C}$.

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$$\tau \circ \phi(f) = \tau \circ \psi(f) \text{ for all } f \in C(X), \quad (e\,0.9)$$
$$\phi^\ddagger = \psi^\ddagger \text{ and} \quad (e\,0.10)$$
$$\overline{R_{\phi,\psi}} = 0. \quad (e\,0.11)$$