Abstract regular polytopes acting as transitive subgroups of Sym(n)
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Happy Valentine’s Day 2015
Platonic Solids

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Outline

- Introduction
  - Regular polytopes and string C-groups
  - String C-groups acting on sets
  - Permutation graphs
- Fracture Graphs
- Ideas and Results in High Ranks
[5, 3] = \langle \rho_0, \rho_1, \rho_2 \mid \rho_i^2 = 1 = (\rho_0 \rho_2)^2 = (\rho_0 \rho_1)^5 = (\rho_1 \rho_2)^3 \rangle
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[3, 4] = \langle \rho_0, \rho_1, \rho_2 \mid \rho_i^2 = 1 = (\rho_0 \rho_2)^2 = (\rho_0 \rho_1)^3 = (\rho_1 \rho_2)^4 \rangle

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Let $\Gamma$ be a (string C) group, and let it act on the set $[1, \ldots, n]$. 

This makes $n$ really big. 

$[1, \ldots, n]$ = orbits of flags of the polytope. 

This makes $n$ small, but we lose the group structure in the graph.
String C-groups acting

Let $\Gamma$ be a (string C) group, and let it act on the set $[1, \ldots, n]$.

- $[1, \ldots, n] = \text{flags of the polytope or elements of } \Gamma$
  - Maniplexes
  - Monodromy
  - Caley Graphs
  - Colorful Polytopes

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Let \( n \) be as small as possible so that the action of \( \Gamma \) on \([1, \ldots, n]\) is faithful.

A string \( C \)-group \( \Gamma \) will be of “high rank” if its rank is “close” to \( n \).
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A string $\mathbb{C}$-group $\Gamma$ will be of “high rank” if its rank is “close” to $n$.

Example 1: High rank

Let $\Gamma$ be the automorphism group of the regular tetrahedron $[3, 3]$. Then $r = 3$ and $n = 4$ where $[1, \ldots, n]$ can be the vertices of the tetrahedron.
Let \( n \) be as small as possible so that the action of \( \Gamma \) on \([1, \ldots, n]\) is faithful.

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Example 2: High rank

Let \( \Gamma \) be the automorphism group of the regular cube \([4, 3]\). Then \( r = 3 \) and \( n = 6 \) where \([1, \ldots, n]\) can be the 2-faces of the cube.

Since \( \Gamma \cong \text{Sym}(4) \times 2 \) no smaller \( n \) will work.
Let $n$ be as small as possible so that the action of $\Gamma$ on $[1, \ldots, n]$ is faithful.

A string C-group $\Gamma$ will be of “high rank” if its rank is “close” to $n$.

**Example 3: Not high rank**

Let $\Gamma$ be the O’Nan sporadic group.
Then $r = 4$ and $n = 122760$. 
Let $\Gamma$ be a permutation group of degree $n$ generated by involutions $\rho_0, \ldots, \rho_{r-1}$.

The graph $X$ with vertices $[1, \ldots, n]$ and $\{a, b\} \in E(X) \iff a\rho_i = b$ is called the permutation representation graph of $\Gamma$.

Note 1: This is not a new idea. Marston had already been studying “Schreier coset graphs and their applications” in 1992.

Note 2: These are edge labeled multigraphs. Where each transposition in the involution $\rho_i$ gives one $i$-edge.
Symmetries of a cube acting on faces
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Define $F$ as follows from a permutation representation graph $X$ with all $\Gamma_i$ intransitive.

- $V(F) = V(X)$
- $|E(F)| = r$ where $r$ is the rank of $\Gamma$
- $\{a, b\} \in E(F) \Rightarrow a \rho_i = b \Rightarrow a \rho_i \neq b$ for all $\rho_i \in \Gamma_i$
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Fracture Graphs
Lemma:

Let $F$ be a fracture graph and $X$ be a permutation representation graph for a group of degree $n$ generated by $r$ involutions.

- $F$ contains no cycles.
- $F$ has $n - r$ connected components.
- If there is a multi-edge in $X$ then the vertices are in different connected components of $F$.
- If there are two $i$-edges in $X$ then all the vertices of these edges are not in the same connected component of $F$. 
High Rank Fracture Graphs

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Let $r = n - 1$. $F$ is a tree. $X$ has no multi-edges. There are not two $i$-edges in $X$ for any $i$. 

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Additionally, if $(\rho_i \rho_j)^2$ for $|i - j| > 1$

then $\Gamma \cong \text{Sym}(n)$ and $\langle \rho_0, \ldots, \rho_{r-1} \rangle$ gives the Coxeter generators of the $n - 1$ simplex
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then \( \Gamma \cong \text{Sym}(n) \) and \( \langle \rho_0, \ldots, \rho_{r-1} \rangle \) gives the Coxeter generators of the \( n - 1 \) simplex, if all \( \Gamma_i \) are intransitive.
Abstract regular polytopes acting as transitive subgroups of $S_n$

**Theorem:** Cameron, Fernandes, Leemans, M

Let $\Gamma$ be a string $C$-group of rank $r$ which is isomorphic to a transitive subgroup of $S_n$ other than $S_n$ or $A_n$. Then one of the following holds:

1. $r \leq n/2$;
2. $n \equiv 2 \mod 4$, $r = n/2 + 1$ and $\Gamma$ is $2 \wr S_{n/2}$. The Schläfli type is $[2, 3, \ldots, 3, 4]$.
3. $n = 6, 8$, and $\Gamma$ is one of four imprimitive examples.
4. $n = 6$, and $\Gamma$ is $PGL_2(5) \cong S_5$ (the 4-simplex).
Thanks and Happy Birthday!!!