A Spiky Ball

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Special Case: The Illumination Problem

Gohberg–Markus–Levi–Boltyanskii–Hadwiger Illumination Conjecture

Fix $n$. Then the maximum of $N(K, \text{int } K)$ over all convex bodies $K$ in $\mathbb{R}^n$ is $2^n$, and only attained by parallelotopes.

Known:
If $K$ is smooth then $i(K) = n + 1$.

Rogers

$$i(K) := N(K, \text{int } K) \leq \begin{cases} 2^n(n \ln n + n \ln \ln n + 5n) & \text{if } K = -K, \\ \binom{2n}{n}(n \ln n + n \ln \ln n + 5n) & \text{otherwise}. \end{cases}$$
Main Result

Theorem

Let $1 < D < 1.116$ be given. Then for any, sufficiently large dimension $n$, there is an $o$-symmetric convex body $K$ in $\mathbb{R}^n$, with illumination number

$$i(K) = N(K, \text{int } K) \geq .05D^n,$$

for which

$$\frac{1}{D} B^n \subset K \subset B^n.$$
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Let $1 < D < 1.116$ be given. Then for any, sufficiently large dimension $n$, there is an $o$-symmetric convex body $K$ in $\mathbb{R}^n$, with illumination number

$$i(K) = N(K, \text{int } K) \geq .05D^n,$$

for which

$$\frac{1}{D} B^n \subset K \subset B^n.$$  \hspace{1cm} (1)

**Sharp:**

If $\frac{1}{D} B^n \subset K \subset B^n$ for some $D > 1$, then

$$i(K) \leq \frac{n \ln n + n \ln \ln n + 5n}{\Omega_{n-1}(\alpha)},$$

where $\alpha = \arcsin(1/D)$.  \hspace{1cm} (3)
Application: Gap between ill and vein

\( K = -K \)

*Illumination parameter* [K. Bezdek ’06]:

\[
\text{ill}(K) = \inf \left\{ \sum_{p \in \text{vert} P} \|p\|_K \mid P \text{ a polytope such that } \text{vert} P \text{ illuminates } K \right\}.
\]

*Vertex index* [K. Bezdek – A. Litvak ’07]:

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\text{vein}(K) = \inf \left\{ \sum_{p \in \text{vert} P} \|p\|_K \mid P \text{ a polytope such that } K \subseteq P \right\}.
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\[ \text{ill}(K) \geq \text{vein}(K), \quad \text{and} \quad \text{equal for smooth bodies.} \]

[B-L ’07, Gluskin – L. ’12]: \( \text{vein}(\mathcal{B}^n) \) is of order \( n^{3/2} \).
Application: Gap between \( \text{ill} \) and \( \text{vein} \)

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*Let \( K \) be a spiky ball.* Then

\( \text{vein}(K) \) is of order \( n^{3/2} \),

\( \text{ill}(K) \geq i(K) \) is exponentially large.
Preliminaries

$u \in \mathbb{S}^{n-1}$, and $0 < \varphi < \pi/2$.

**Spherical cap:** $C(u, \varphi) = \{ v \in \mathbb{S}^{n-1} : \angle(u, v) \leq \varphi \}$.

**Probability measure of cap:** $\Omega_{n-1}(\varphi)$.

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**Lemma (Böröczky – Wintsche '03)**

Let $0 < \varphi < \pi/2$. Then

\[
\Omega_n(\varphi) > \frac{\sin^n \varphi}{\sqrt{2\pi(n+1)}},
\]

\[
\Omega_n(\varphi) < \frac{\sin^n \varphi}{\sqrt{2\pi n} \cos \varphi}, \quad \text{if } \varphi \leq \arccos \frac{1}{\sqrt{n+1}};
\]

\[
\Omega_n(t\varphi) < t^n \Omega_n(\varphi), \quad \text{if } 1 < t < \frac{\pi}{2\varphi}.
\]

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Roughly,

\[ \Omega_n(\varphi) \approx \sin^n \varphi. \]
The construction

$X_1, \ldots, X_N$ independent random points on $S^n$.

$$K = \text{conv} \left( \{ \pm X_i : i \in [N] \} \cup \frac{1}{D}B^{n+1} \right).$$

Clearly, $K$ is $o$-symmetric and $\frac{1}{D}B^{n+1} \subset K \subset B^{n+1}$.

Need: illumination number not small.
Bad event $E_1$

Notation: $\frac{\pi}{4} < \alpha < \frac{\pi}{2}$ is such that $\sin \alpha = 1/D$.

$E_1$

The event that there are $i \neq j \in [N]$ with $\angle(X_i, X_j) < \pi - 2\alpha$ or $\angle(-X_i, X_j) < \pi - 2\alpha$.

If $E_1$ does not occur, then

for all $i \in [N]$: the set of directions that illuminate $K$ at $X_i$ is the spherical cap $C(-X, \alpha)$.
Bad event $E_2$

$T \in \mathbb{Z}^+$ fixed.

$E_2$

There is a direction $u \in \mathbb{S}^n$ with $|C(u, \alpha) \cap \{\pm X_i : i \in [N]\}| > T$.

If NOT($E_1$) AND NOT($E_2$), then

$$i(K) \geq 2N / T.$$
Bad event $E_2$

$T \in \mathbb{Z}^+ \text{ fixed.}$

<table>
<thead>
<tr>
<th>$E_2$</th>
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**If NOT($E_1$) AND NOT($E_2$), then**

$$i(K) \geq 2N/T.$$  

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<td>$\mathbb{P}(E_2)$ is hard to estimate.</td>
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$E'_2$: finitize $E_2$

$E_2$
There is a direction $u \in \mathbb{S}^n$ with $|C(u, \alpha) \cap \{\pm X_i : i \in [N]\}| > T$.

Fix $\delta > 0$.
Let $\Lambda$ be a $\delta$-net of $\mathbb{S}^n$.
Let $p = 2\Omega_n(\alpha + \delta)$.
Let $\Theta > 1$ be fixed, and set $T = N\Theta p$.

$E'_2$
There is a direction $v \in \Lambda$ with $|C(v, \alpha + \delta) \cap \{\pm X_i : i \in [N]\}| > N\Theta p$.

Clearly, if $E_2 \implies E'_2$.

If $\text{NOT}(E_1) \text{ AND } \text{NOT}(E'_2)$, then

$$i(K) \geq 2/(\Theta p).$$
Is it possible?

If \( \neg E_1 \) AND \( \neg E_2' \), then

\[ i(K) \geq \frac{2}{(\Theta p)}. \]

The task

Need to set \( N, \Theta, \delta \) such that \( P(\neg E_1 \text{ and } \neg E_2') > 0 \) and \( 2/(\Theta p) \) is exponentially large in \( n \).
Is it possible?

If $\text{NOT}(E_1) \text{ AND NOT}(E_2')$, then

$$i(K) \geq \frac{2}{(\Theta p)}.$$ 

The task

Need to set $N, \Theta, \delta$ such that $\mathbb{P}(\text{not}(E_1) \text{ and not}(E_2')) > 0$ and $\frac{2}{(\Theta p)}$ is exponentially large in $n$.

$E_1$

The event that there are $i \neq j \in [N]$ with $\angle(X_i, X_j) < \pi - 2\alpha$ or

$$\angle(-X_i, X_j) < \pi - 2\alpha.$$ 

Equation 1

$$\mathbb{P}(E_1) \leq N^2 \Omega_n (\pi - 2\alpha) \leq 1/4.$$
How to set $N, \Theta, \delta$

$E'_2$

There is a direction $\nu \in \Lambda$ with $|C(\nu, \alpha + \delta) \cap \{\pm X_i : i \in [N]\}| > N\Theta p$.

Fix $\nu \in \Lambda$.

When $X_i$ is picked randomly, the probability that $\nu$ is contained in $C(X_i, \alpha + \delta)$ or in $C(-X_i, \alpha + \delta)$ is $p = 2\Omega_n(\alpha + \delta)$. Thus,

$$\mathbb{P}(E'_2) \leq |\Lambda| \mathbb{P}(\xi > N\Theta p) \leq 1/4 \quad \text{with } \xi \sim \text{Binom}(N, p).$$
How to set $N, \Theta, \delta$

$E'_2$

There is a direction $v \in \Lambda$ with $|C(v, \alpha + \delta) \cap \{\pm X_i : i \in [N]\}| > N\Theta p$.

Fix $v \in \Lambda$.
When $X_i$ is picked randomly, the probability that $v$ is contained in $C(X_i, \alpha + \delta)$ or in $C(-X_i, \alpha + \delta)$ is $p = 2\Omega_n(\alpha + \delta)$. Thus,

$$P(E'_2) \leq |\Lambda| P(\xi > N\Theta p) \leq 1/4 \text{ with } \xi \sim \text{Binom}(N, p).$$

Easy: There is a $\Lambda$ with $|\Lambda| \leq n^2 / \sin^n(\delta)$.

Equation 2

$$P(E'_2) \leq \frac{n^2}{\sin^n(\delta)} P(\xi > N\Theta p) \leq 1/4 \text{ with } \xi \sim \text{Binom}(N, p).$$
Can we set $N, \Theta, \delta$ properly?

**Equation 1**

\[ \mathbb{P}(E_1) \leq N^2 \Omega_n(\pi - 2\alpha) \leq 1/4. \]

**Equation 2**

\[ \mathbb{P}(E_2') \leq \frac{n^2}{\sin^n(\delta)} \mathbb{P}(\xi > N\Theta p) \leq 1/4 \quad \text{with } \xi \sim \text{Binom}(N, p). \]

**Equation 3**

Make $2/(\Theta p)$ exponentially large.
Can we set $N, \Theta, \delta$ properly?

**Equation 1**

$$\mathbb{P}(E_1) \leq N^2 \Omega_n(\pi - 2\alpha) \leq 1/4.$$ 

**Equation 2**

$$\mathbb{P}(E'_2) \leq \frac{n^2}{\sin^n(\delta)} \mathbb{P}(\xi > N\Theta p) \leq 1/4 \quad \text{with} \quad \xi \sim \text{Binom}(N, p).$$

**Equation 3**

Make $2/(\Theta p)$ exponentially large.

Yes, we can!
Good event: Happy Birthday, Karcsi and Egon!