Cubic Irrationalities and a Ramanujan-Nagell Analogue

Mark Bauer

* Joint work with Michael Bennett

University of Calgary
Department of Mathematics & Statistics

Motivating Question

Does the equation

$$x^3 - x + 8 = 2^k$$

have any solutions for $x > 8$?

<u>Note:</u> It has solutions for $x \in \{-2, -1, 0, 1, 3, 5, 8\}$.

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Better(?) Question: Let $f(x) \in \mathbb{Z}[x]$ be a monic cubic. How many solutions can the following equation have?

$$f(x) = 2^k$$

For $n \in \mathbb{Z}$, define $P(n) = p$ where p is the largest prime s.t. $p|n$.

$$P(f(x)) \geq c \cdot \log\log\max\{|x|, 3\}$$

There is also a homogenized form of this.

Thue-Mahler Equation

- $F(x,y) \in \mathbb{Z}[x,y]$ irreducible binary form of degree at least 3.
- $\{p_1, \ldots p_s\}$ be a finite set of primes.

The Thue-Mahler equation is given by

$$F(x,y) = p_1^{a_1} \cdots p_s^{a_s}.$$

$$F(x,y) = 2^k$$

Evertse: Number of solutions has explicit upper bound

We have an obvious simplification (for $a, b \in \mathbb{Z}$).

$$x^3 + ax + b = 2^k \qquad \text{or} \qquad x^3 + ax + b = 27 \cdot 2^k$$

Note: this says

$$|x^3 - 2^n| = |ax + b|$$

or

$$|x^3 - 27 \cdot 2^n| = |ax + b|$$

i.e. the difference of a cube and a power of 2 is "small".

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We'll drop the one with 27 from here one....

Another approach to finiteness:

Show

$$|x^3 - 2^n| \geq x^\delta, \ \delta > 1.$$

Should we expect this?

- $3|n$: then its either 0 or $> x^2$.
- $3 \not| n$: yields an approximation to a power of $\sqrt[3]{2}$.

More precisely, $n = 3k + b$, $k \in \mathbb{Z}$, $b \in \{1, 2\}$.
Our top inequality is (roughly) equivalent to

$$\left| \sqrt[3]{2^b} - \frac{x}{2^k} \right| \gg \frac{1}{2^{k(3-\delta)}}$$

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So?

Irrationality Measures

Explicit Irrationality Measure

Let α be a real number. If $C_\alpha, \lambda_\alpha \in \mathbb{R}$ such that for all $\alpha \neq p/q \in \mathbb{Q}$:

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{C_\alpha}{q^{\lambda_\alpha}}$$

then C_α, λ_α is an explicit irrationality measure for α.

If we have $\lambda_{\sqrt[3]{2}} < 3 - \delta$, then

$$|x^3 - 2^n| \geq x^\delta.$$

with finitely many exceptions.

Note: We need $\delta > 1$, so that means we want $\lambda_{\sqrt[3]{2}} < 2$.

Theorem (Liouville's Theorem)

$\lambda_\alpha = [\mathbb{Q}[\alpha] : \mathbb{Q}],\ C_\alpha = C(\alpha).$

Theorem (Thue (1918))

$\lambda_\alpha = n/2 + 1 + \epsilon,\ C_\alpha = C(\alpha, \epsilon)$

Theorem (Siegel (1921))

$\lambda_\alpha = \min_{s \in 1,\ldots,n-1}(\frac{n}{s+1} + s) \approx 2\sqrt{n},\ C_\alpha = C(\alpha)$

Theorem (Gelfond-Dyson (1947))

$\lambda_\alpha = \sqrt{2n} + \epsilon,\ C_\alpha = C(\alpha, \epsilon)$

Theorem (Roth (1955))

$\lambda_\alpha = 2 + \epsilon,\ C_\alpha = C(\alpha, \epsilon)$

Restricted Irrationality Measures

Note: Our motivating example give rise to considering rational approximations with denominators of a specific form:

$$\left|\sqrt[3]{2} - \frac{x}{2^k}\right| < \frac{1}{2^{k(3-\delta)}}, \quad 3 - \delta < 2.$$

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Explicit Restricted Irrationality Measure

Let α be a real number, $y \in \mathbb{Z}^+$. If $C_{\alpha,y}, \lambda_{\alpha,y} \in \mathbb{R}$ such that for all $\alpha \neq p/y^k$:

$$\left|\alpha - \frac{p}{y^k}\right| \geq \frac{C_{\alpha,y}}{(y^k)^{\lambda_{\alpha,y}}}$$

then $C_{\alpha,y}, \lambda_{\alpha,y}$ is a restricted irrationality measure for α.

Restricted Irrationality Measures

Theorem (Ridout's Theorem (simplified))

Let α be an algebraic integer (non-rational), and $\{Q_1, \ldots, Q_n\}$ be a fixed set of primes. Let q be any integer with prime factors in $\{Q_1, \ldots Q_n\}$. The inequality

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{1+\epsilon}}$$

has at most finitely many solutions in p and q for every $\epsilon > 0$.

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1. This is good enough.
2. Still ineffective $\Rightarrow$ We want **effective** version of this theorem.

Towards Constructing Restricted Irrationality Measures

Thue-Siegel Method

1. Find an initial good rational approximation.
2. Generate an infinite family of "good" approximations using the initial one.
3. Use this family to construct the irrationality measure.

Obviously, we will cover this in reverse order......

Constructing Irrationality Measures

Construct an infinite sequence of rationals $\{P_k/Q_k\}_k$ s.t.

$$\frac{P_k}{Q_k} \approx \sqrt[3]{2}$$

with $Q_k \to \infty$, with Q_k "sufficiently" dense.

Assume $\frac{p}{q} \in \mathbb{Q}$ is also close to $\sqrt[3]{2}$.

$$\left|\sqrt[3]{2} - \frac{P_k}{Q_k}\right| + \left|\sqrt[3]{2} - \frac{p}{q}\right| \geq \left|\frac{P_k}{Q_k} - \frac{p}{q}\right| \geq \frac{1}{L_k}$$

where $L_k = lcm(q, Q_k)$, provided $\frac{P_k}{Q_k} \neq \frac{p}{q}$. Taking

$$\left|\sqrt[3]{2} - \frac{P_k}{Q_k}\right| \leq \frac{1}{2\,L_k}, \quad \Longrightarrow \quad \left|\sqrt[3]{2} - \frac{p}{q}\right| \geq \frac{1}{2\,L_k}$$

Hypergeometric Method

Padé Approximants to $(1-z)^{1/3}$:

$$I_{n_1,n_2}(x) = P_{n_1,n_2}(x) - (1-x)^{1/3} Q_{n_1,n_2}(x)$$

where $I_{n_1,n_2}(x) = O\left(x^{n_1+n_2+1}\right)$.

$$P_{n_1,n_2}(x) = \sum_{k=0}^{n_1} \binom{n_2+1/3}{k} \binom{n_1+n_2-k}{n_2} (-x)^k$$

$$Q_{n_1,n_2}(x) = \sum_{k=0}^{n_2} \binom{n_1-1/3}{k} \binom{n_1+n_2-k}{n_2} (-x)^k$$

1. $I_{n_1,n_2}(x)$ also has a “nice” integral representation.
2. Can approximate $I_{n_1,n_2}(x), P_{n_1,n_2}(x)$ for “small” x analytically.

Hypergeometric Method

Start with a "good" approximation:

$$\frac{5}{2^2} \approx 2^{1/3} \quad \Rightarrow \quad 5^3 + 3 = 2^7, \quad 3 \text{ is "small"}.$$

Then

$$\frac{Q_{n_1,n_2}}{P_{n_1,n_2}}\left(\frac{3}{2^7}\right) \approx \left(1 - \frac{3}{2^7}\right)^{-1/3} = \frac{\sqrt[3]{2} \cdot 2^2}{5}$$

$$\Longrightarrow \frac{Q_{n_1,n_2}}{P_{n_1,n_2}}\left(\frac{3}{2^7}\right) \approx \frac{\sqrt[3]{2} \cdot 2^2}{3}$$

Simplifying appropriately we get P_k and Q_k.

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Simplifying appropriately we get P_k and Q_k.

Note: The denominator of the left is divisible by $(2^7)^{n_2 - n_1}$.

Point: One "good" approximation gives us many.

Theorem

For $k \geq 12$, and $s \in \{1,3\}$, we have for all $p \in \mathbb{Z}$

$$\left| \sqrt[3]{2^b} - \frac{p}{s \cdot 2^k} \right| > \left(2^k\right)^{-1.63}.$$

Corollary

If x and n are integers, then either $x^3 = 2^n$, $(x,n) = (5,7)$ or we have

$$\left| x^3 - 2^n \right| \geq x^{4/3}.$$

If x and n are integers, then either $x^3 = 27 \cdot 2^n$, or $x \in \{4,5,8,15,19,38,121\}$ or we have

$$\left| x^3 - 27 \cdot 2^n \right| \geq 3^{5/3} x^{4/3}.$$

Two Applications

1. Our original equation (exclude $x = 5$),

$$x^3 - x + 8 = 2^n$$

$$\Rightarrow |x^3 - 2^n| = |x - 8| \geq x^{4/3} \quad \text{or} \quad x^3 = 2^n.$$

This inequality is only satisfied for $x < 3$.

2. Cubic Ramanujan-Nagell Analogue (exclude $x = 5$):

$$x^3 + D = 2^n$$

$$\Rightarrow |x^3 - 2^n| = |D| \geq x^{4/3} \quad \text{or} \quad x^3 = 2^n$$

Again, interested in number of solutions.

- Finite is easy.
- At most 3 - gap principle