Distance between classes of S^1-valued maps

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BIRS Workshop - New Trends in Nonlinear Elliptic Equations

Examples of classes of S^1-valued Sobolev maps

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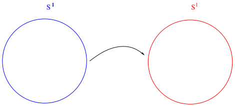

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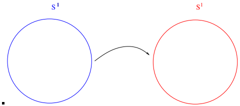

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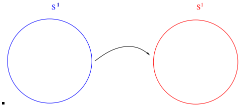

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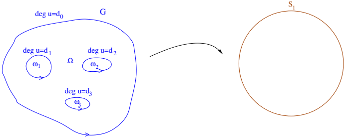

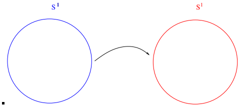

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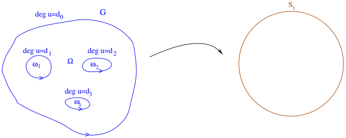

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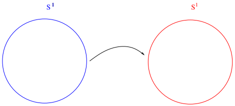

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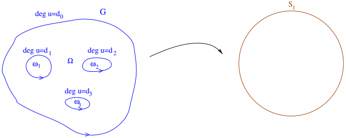

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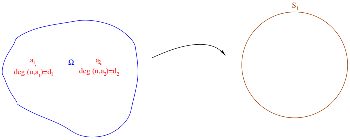

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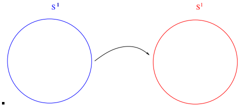

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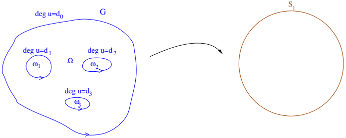

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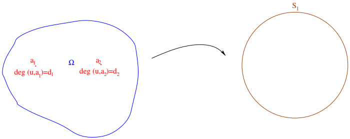

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- White, Bethuel, Brezis-Nirenberg, Brezis-Y.Y.Li, Hang-Lin,. . .

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Theorem (Rubinstein-Sh 07)

$\delta_p(d_1, d_2) = \frac{2^{1+1/p}}{\pi^{1-1/p}} |d_2 - d_1|$.

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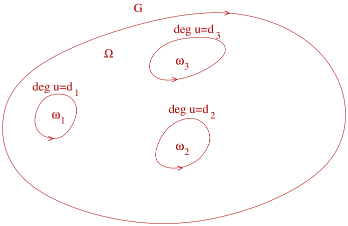

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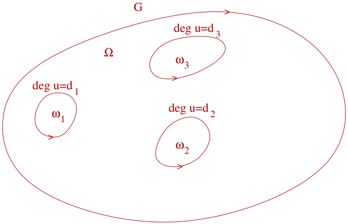

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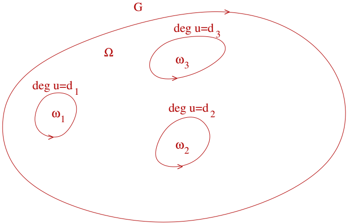

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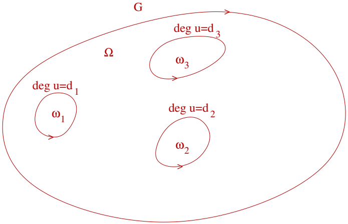

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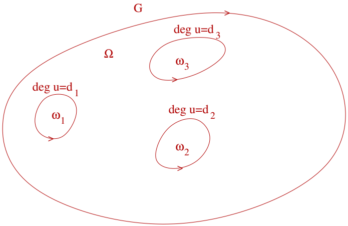

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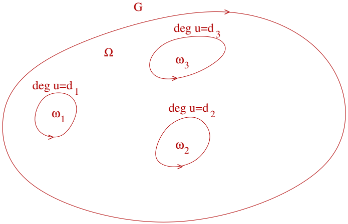

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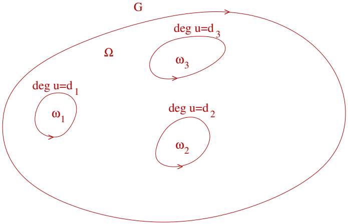

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- In many cases equality holds in (1), but not always!

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(Bethuel, Demengel, Brezis-Mironescu-Ponce, Brezis-Mironescu,. . .)

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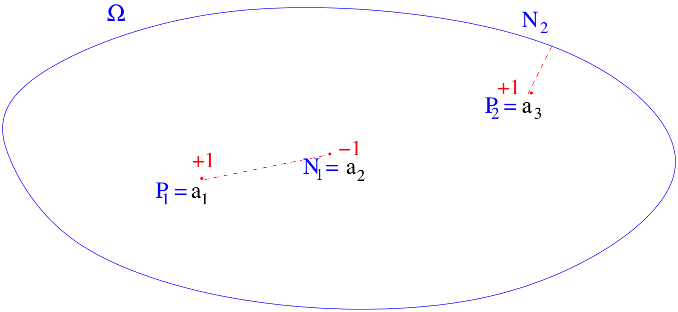

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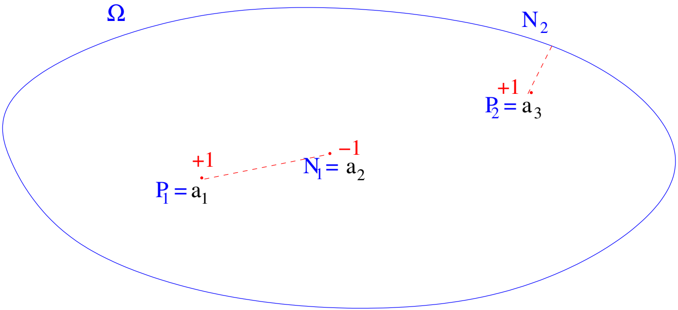

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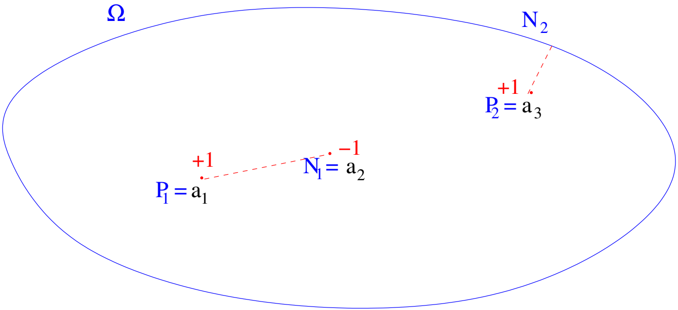

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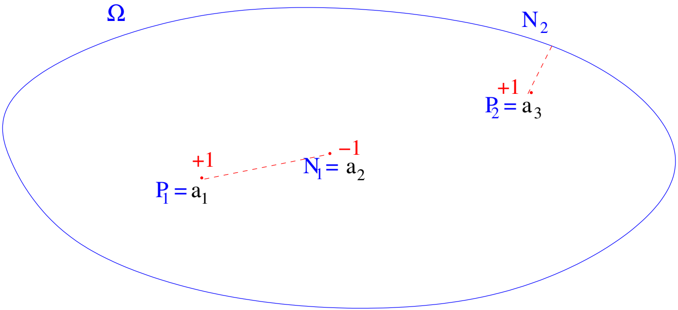

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- $u \sim v$ iff $Ju = Jv$ where $Ju := \frac{1}{2}\operatorname{curl}(u \wedge \nabla u)$ (a distribution).

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- In particular,

$d(\mathcal{E}(u_0), \mathcal{E}(v_0)) = \frac{2}{\pi}\Sigma(u_0\bar{v}_0)$ and $d_H(\mathcal{E}(u_0), \mathcal{E}(v_0)) = \Sigma(u_0\bar{v}_0)$.

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This yields the lower bound in all statements.

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- Use (2) to choose $\{\zeta_n\} \subset S^1$ s.t.

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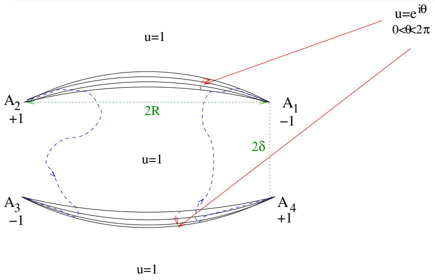

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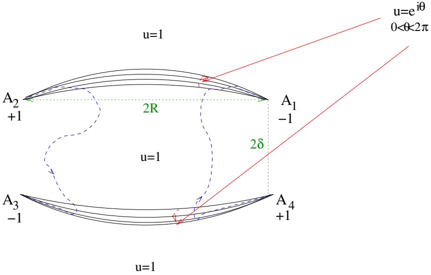

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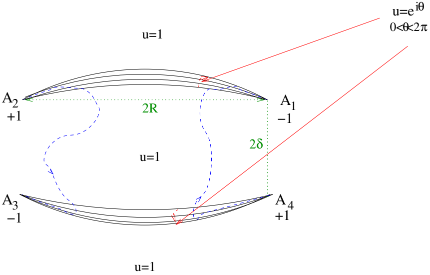

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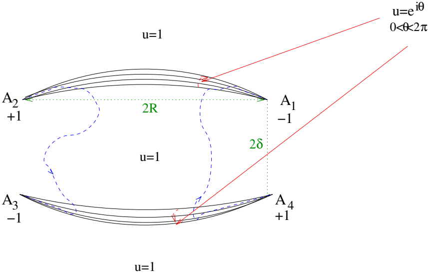

- Assume $\delta \ll R \ll \mathrm{dist}(\{A_j\}_{j=1}^4, \partial\Omega)$ $(\Sigma(u) = (2\pi)\cdot 4\delta)$.
- Let $v \in C^\infty(\bar{\Omega}; S^1)$, $w = u\bar{v}$, $\mathcal{R} \subset S^1$ the regular values of w:

$$\int_\Omega |\nabla(u-v)| = \int_{\zeta\in\mathcal{R}} \int_{[w=\zeta]} \frac{|\nabla(u-v)|}{|\nabla w|} \geq \cdots \geq 8\pi\delta(1 - O(\delta/R)).$$

Thank you for your attention!