Real Galois cohomology of semisimple groups

Mikhail Borovoi, Tel Aviv University

Joint work with Dmitry A. Timashev

Banff, September 14, 2015

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$$\mathrm{SO}(n) = \mathrm{SO}(n, \mathbb{R}) = \{g \in \mathrm{GL}(n, \mathbb{R}) \mid g^t I_n \, g = I_n, \ \det(g) = 1\}$$

where $I_n = \mathrm{diag}(1, \dots, 1)$

$$\mathrm{SO}(n-2, 2) = \{g \in \mathrm{GL}(n, \mathbb{R}) \mid g^t I_{n-2,2} \, g = I_{n-2,2}, \ \det(g) = 1\}$$

where $I_{n-2,2} = \mathrm{diag}(1, \dots, 1, -1, -1)$
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Real algebraic groups

Let G be a real algebraic group. We write $G(\mathbb{R})$ for G.
$G \subset GL(n, \mathbb{R})$ is defined by polynomial equations with real coefficients.
$G(\mathbb{C}) \subset \mathrm{GL}(n, \mathbb{C})$ is the group of complex solutions of those equations.
We have an antiholomorphic involutive automorphism of complex conjugation

$$\rho\colon G(\mathbb{C}) \to G(\mathbb{C}), \quad \rho^2 = \mathrm{id}.$$

Then $G(\mathbb{R}) = \{g \in G(\mathbb{C}) \mid \rho(g) = g\}$.

Examples:

$G(\mathbb{R}) = \mathrm{SO}(n),\ G(\mathbb{C}) = \mathrm{SO}(n, \mathbb{C}), \rho(g) = \bar{g}$
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$H^1(\mathbb{R}.G)$

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Def. The group $G(\mathbb{C})$ acts on the right on $Z^1(\mathbb{R}, G)$ by

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hence g is a cocycle, i.e. $g \in Z^1(\mathbb{R}, G)$, and we obtain a cohomology class $[g] \in H^1(\mathbb{R}, G)$.

If G is connected (over $\mathbb{C}$), then any $\xi \in H^1(\mathbb{R}, G)$ is of the form $\xi = [g]$ with $g \in G(\mathbb{R})_2$ (Kottwits 1986).

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History

One needs $H^1(\mathbb{R}, G)$ for classification problems. If X is an algebraic variety over $\mathbb{R}$, or a Lie algebra, or an algebraic group etc.,

$$\{\text{Real forms of } X\} / \text{Isom} \longleftrightarrow H^1(\mathbb{R}, \mathrm{Aut}(X)).$$

I consider connected semisimple $\mathbb{R}$-groups. For classical groups, the Galois cohomology $H^1(\mathbb{R}, G)$ is well known. For adjoint groups – Élie Cartan, then F. R. Gantmacher in 1939 and Victor Kac in 1969. For simply connected groups of type $\mathbf{E}_7$ – Skip Garibaldi and Nikita Semenov (2010) and Brian Conrad (2013). For simply connected groups – Jeffrey Adams 2013 and Borovoi-Evenor 2014. In my work with Dmitry Timashev we compute H^1 for any semisimple $\mathbb{R}$-group that is an inner form of a compact group, i.e., has a compact maximal torus.

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Half-spin group

I explain our results by the example of the half-spin group $\mathrm{HSpin}(12)$.

Set $G = \mathrm{SO}(2\ell, \mathbb{R})$. This group admits a universal covering

$$1 \to \Gamma_0 \to \widetilde{G} \to G \to 1,$$

where $\Gamma_0 \simeq \mathbb{Z}/2\mathbb{Z}$, $\widetilde{G} = \mathrm{Spin}(2\ell, \mathbb{R})$.

Set $\Gamma = Z(\widetilde{G})$. We assume that $\ell = 2k$ ($G = \mathrm{SO}(4k)$), Then

$$\Gamma \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

Γ has 3 subgroups of order 2: Γ_0, Γ_1, and Γ_2. We have

$$G = \widetilde{G}/\Gamma_0$$

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$$\mathrm{HSpin}(4k) = \widetilde{G}/\Gamma_1\,.$$

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Groups

I will compute H^1 for:

$G = \mathrm{SO}(12)$,

$\widetilde{G} = \mathrm{Spin}(12)$,

$\mathrm{HSpin}(12)$,

$G^{\rm ad} := \widetilde{G}/\Gamma = \mathrm{PSO}(12) = \mathrm{SO}(12)/\{\pm 1\}$.

Kac 2-labelings

We work with extended Dynkin diagrams. Let $k = 3$, $G = \mathrm{SO}(12)$ of type $\mathbf{D}_6$. The extended Dynkin diagram $\widetilde{D}$ is

Its vertices correspond to roots: the simple roots

$$\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6,$$

and the *lowest root* α_0. These roots are linearly dependent, and coefficients of this dependence are written on the right-hand diagram.

Def. A Kac 2-labeling of $\widetilde{D}$ is $\boldsymbol{p} = (p_j)_{j=1,\ldots,6,0} \in \mathbb{Z}^7_{\geq 0}$ such that

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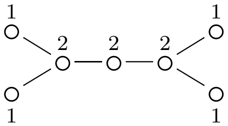

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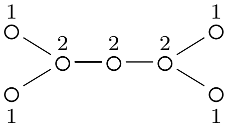

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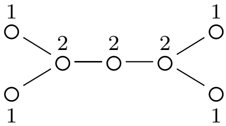

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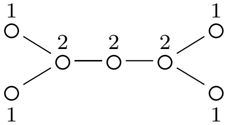

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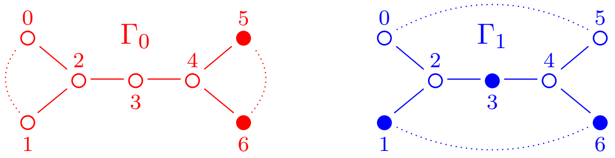
The red group and the blue group

Let $\mathcal{K}_2 = \mathcal{K}_2(\widetilde{D})$ denote the set of Kac 2-labelings.

The group $\Gamma = \Gamma_0 \cdot \Gamma_1$ acts on $\widetilde{D}$ and hence, on $\mathcal{K}_2(\widetilde{D})$:



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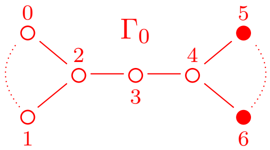

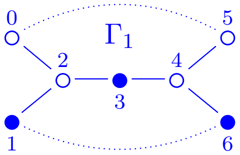

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We have 6 orbits, hence $\#H^1(\mathbb{R}, G^{\mathrm{ad}}) = 6$.

Cocycles

With any $\boldsymbol{q} \in \mathcal{K}_2$ we associate a cocycle $c(\boldsymbol{q}) \in G^{\mathrm{ad}}(\mathbb{R})_2$. We construct $c(\boldsymbol{q})$ as follows:

Let $T^{\mathrm{ad}} \subset G^{\mathrm{ad}}$ be a maximal torus,
$X^*(T^{\mathrm{ad}}_{\mathbb{C}}) = \mathrm{Hom}(T^{\mathrm{ad}}_{\mathbb{C}}, \mathbf{G}_{m,\mathbb{C}})$ the character group,
$R = R(G^{\mathrm{ad}}_{\mathbb{C}}, T^{\mathrm{ad}}_{\mathbb{C}}) \subset X^*(T^{\mathrm{ad}}_{\mathbb{C}})$ the root system,
$\Pi = \{\alpha_1, \dots, \alpha_6\}$ a basis of R.
We define $c = c(\boldsymbol{q}) \in T^{\mathrm{ad}}(\mathbb{R})_2 \subset G^{\mathrm{ad}}(\mathbb{R})_2$ by

$$\alpha_i(c) = (-1)^{q_i} \quad (i = 1, \dots, 6).$$

Twisting

We consider the twisted group ${}_{c(\boldsymbol{q})}G$.

This means that we construct a new $\mathbb{R}$-group G' with $G'(\mathbb{C}) = G(\mathbb{C})$, but with a new complex conjugation

$$\rho' = c(\boldsymbol{q}) \circ \rho,$$

where $c(\boldsymbol{q}) \in G^{\mathrm{ad}}(\mathbb{R})_2 \subset \mathrm{Aut}(G)_2$.

We write $G(\boldsymbol{q}) = {}_{c(\boldsymbol{q})}G := G'$.

Representatives of the orbits of Γ and the corresponding twisted forms

$$\boldsymbol{q} = {}^{2}_{0}\,0\,0\,0\,{}^{0}_{0} \quad G(\boldsymbol{q}) = \mathrm{SO}(12), \qquad \boldsymbol{q} = {}^{1}_{1}\,0\,0\,0\,{}^{0}_{0} \quad G(\boldsymbol{q}) = \mathrm{SO}(10,2),$$

$$\boldsymbol{q} = {}^{0}_{0}\,1\,0\,0\,{}^{0}_{0} \quad G(\boldsymbol{q}) = \mathrm{SO}(8,4), \qquad \boldsymbol{q} = {}^{0}_{0}\,0\,1\,0\,{}^{0}_{0} \quad G(\boldsymbol{q}) = \mathrm{SO}(6,6),$$

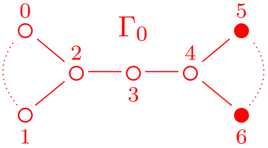
$$\boldsymbol{q} = {}^{1}_{0}\,0\,0\,0\,{}^{1}_{0} \quad G(\boldsymbol{q}) = \mathrm{SO}^*(12), \qquad \boldsymbol{q} = {}^{1}_{0}\,0\,0\,0\,{}^{0}_{1} \quad G(\boldsymbol{q}) = \mathrm{SO}^*(12).$$

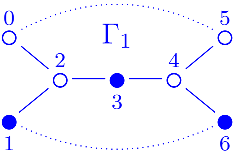
Here $\mathrm{SO}^*(12)$ is the quaternionic form of $\mathrm{SO}(12)$.

Red sum and blue sum

We define

$$\sum_{\textbf{red}}(\boldsymbol{p}) = p_5 + p_6 \mod 2, \qquad \sum_{\textbf{blue}}(\boldsymbol{p}) = p_1 + p_3 + p_6 \mod 2.$$





Thm. $H^1(\mathbb{R}, \mathrm{HSpin}(\boldsymbol{q}))$ in a canonical bijection with the set of orbits of the blue group:

$$H^1(\mathbb{R}, \mathrm{HSpin}(\boldsymbol{q})) \cong \left\{ \boldsymbol{p} \in \mathcal{K}_2 \mid \sum_{\textbf{blue}}(\boldsymbol{p}) = \sum_{\textbf{blue}}(\boldsymbol{q}) \right\} \ / \ \Gamma_1.$$

Similarly for $H^1(\mathbb{R}, G(\boldsymbol{q}))$ with the red sum and the red group Γ_0.

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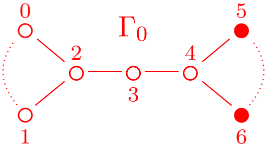

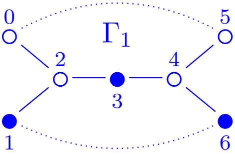

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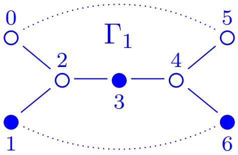
$H^1(\mathbb{R}, \mathrm{HSpin}(12))$

For $\mathrm{HSpin}(12)$ we have $\boldsymbol{q} = {}^{2}_{0}000^{0}_{0}$, hence $\sum_{\textbf{blue}}(\boldsymbol{q}) = 0$.



We have 5 orbits of the blue group Γ_1 with **even** blue sum, hence $\#H^1(\mathbb{R}, \mathrm{HSpin}(12)) = 5$:

$$\left({}^{2}_{0}000^{0}_{0} \quad {}^{0}_{0}000^{2}_{0}\right) \qquad \left({}^{0}_{2}000^{0}_{0} \quad {}^{0}_{0}000^{0}_{2}\right)$$

$$\left({}^{1}_{0}000^{1}_{0}\right) \qquad \left({}^{0}_{1}000^{0}_{1}\right) \qquad \left({}^{0}_{0}100^{0}_{0} \quad {}^{0}_{0}001^{0}_{0}\right)$$

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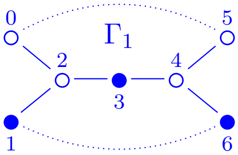

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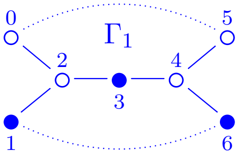
$H^1(\mathbb{R}, \mathrm{HSpin}(10,2))$

For $\mathrm{HSpin}(10,2)$ we have $\boldsymbol{q} = \begin{smallmatrix}1&&&&0\\&0&0&0&\\1&&&&0\end{smallmatrix}$, hence $\sum_{\mathbf{blue}}(\boldsymbol{q}) = 1$.



We have 3 orbits of the blue group Γ_1 with **odd** blue sum, hence $\#H^1(\mathbb{R}, \mathrm{HSpin}(10,2)) = 3$:

$$\left(\begin{smallmatrix}1&&&&0\\&0&0&0&\\0&&&&1\end{smallmatrix} \quad \begin{smallmatrix}0&&&&1\\&0&0&0&\\1&&&&0\end{smallmatrix}\right) \qquad \left(\begin{smallmatrix}1&&&&0\\&0&0&0&\\1&&&&0\end{smallmatrix} \quad \begin{smallmatrix}0&&&&1\\&0&0&0&\\0&&&&1\end{smallmatrix}\right)$$

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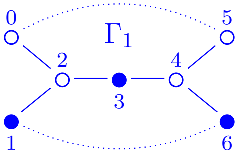

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$\mathrm{HSpin}^*(12)$?

Set

$$\boldsymbol{q}' = {}^{1}_{0}0\,0\,0^{1}_{0} \qquad \boldsymbol{q}'' = {}^{1}_{0}0\,0\,0^{0}_{1}$$

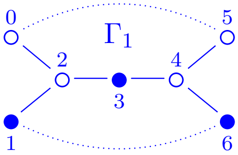
Then

$$\mathrm{SO}(\boldsymbol{q}') =: \mathrm{SO}^*(12) \simeq \mathrm{SO}(\boldsymbol{q}'').$$

However,

$$\mathrm{HSpin}(\boldsymbol{q}') \not\simeq \mathrm{HSpin}(\boldsymbol{q}'').$$

Indeed, for $\boldsymbol{q}'$ the blue sum is 0, hence $\#H^1(\mathbb{R}, \mathrm{HSpin}(\boldsymbol{q}')) = 5$, while for $\boldsymbol{q}''$ the blue sum is 1, hence $\#H^1(\mathbb{R}, \mathrm{HSpin}(\boldsymbol{q}'')) = 3$.



SO(12)

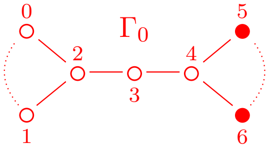
Similarly, using the red group Γ_0 and the red coloring, we obtain

$$\#H^1(\mathbb{R}, \mathrm{SO}(12-2m, 2m)) = 7 \quad m = 0, 1, 2, 3,$$

$$\#H^1(\mathbb{R}, \mathrm{SO}^*(12)) = 2.$$

This is well known and can be obtained elementarily.

For $\mathrm{SO}(12)$



$$\begin{pmatrix} {}^{2}_{0}\,0\,0\,0\,{}^{0}_{0} & & {}^{0}_{2}\,0\,0\,0\,{}^{0}_{0} \end{pmatrix}$$

$$ {}^{1}_{1}\,0\,0\,0\,{}^{0}_{0} \qquad {}^{0}_{0}\,1\,0\,0\,{}^{0}_{0} \qquad {}^{0}_{0}\,0\,1\,0\,{}^{0}_{0} \qquad {}^{0}_{0}\,0\,0\,1\,{}^{0}_{0} \qquad {}^{0}_{0}\,0\,0\,0\,{}^{1}_{1} $$

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We have 7 even orbits, hence $\#H^1(\mathbb{R}, \mathrm{SO}(12)) = 7$

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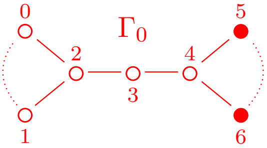

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We have 7 even orbits, hence $\#H^1(\mathbb{R}, \mathrm{SO}(12)) = 7$.

$SO(12)$ as an automorphism group

Elementarily, we have

$$\mathrm{SO}(12) = \mathrm{Aut}(V, \omega, F),$$

where $V = \mathbb{R}^{12}$, ω the standard volume form, and F is the quadratic form with matrix $S = S_F = I$. Thus $H^1(\mathbb{R}, \mathrm{SO}(12))$ classifies real forms of (V, ω, F), i.e., $\mathbb{R}$-isomorphism classes of triples (V', ω', S') that are isomorphic to (V, ω, F) over $\mathbb{C}$.

If $\dim(V') = \dim(V)$, we can choose an isomorphism $(V', \omega') \xrightarrow{\sim} (V, \omega)$, then we need only to classify quadratic forms F' on V over $\mathbb{R}$ with matrix $S' = S_{F'}$ such that

$$S' = g_{\mathbb{C}}^{t}\, S\, g_{\mathbb{C}} \quad \text{for some } g_{\mathbb{C}} \in \mathrm{SL}(12, \mathbb{C}).$$

This is the same as to say that $\det(S') = \det(S) = 1$.

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Symmetric matrices with $\det = 1$

Then we must determine

$$\{S' \in \mathrm{Sym}(12, \mathbb{R}) \mid \det(S') = 1\} \ / \ \mathrm{SL}(12, \mathbb{R}),$$

where

$$S' * g = g^t \, S' \, g, \quad g \in \mathrm{SL}(12, \mathbb{R}).$$

By Sylvester's inertia law we have 7 orbits, the orbits with $2k$ minuses, $k = 0, 1, \ldots, 6$. Thus

$$\# H^1(\mathbb{R}, \mathrm{SO}(12)) = 7.$$

We used only the fact that $\det(S) = 1$, so we obtain the same set of orbits and hence the same cardinality also for $\mathrm{SO}(10, 2)$, $\mathrm{SO}(8, 4)$, and $\mathrm{SO}(6, 6)$.

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$H^1(\mathbb{R}, \mathrm{Spin}(\boldsymbol{q}))$

For the **simply connected** group $\widetilde{G}(\boldsymbol{q}) = \mathrm{Spin}(\boldsymbol{q})$ we have

Thm.

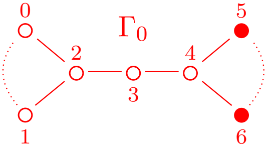
$$H^1(\mathbb{R}, \mathrm{Spin}(\boldsymbol{q})) \cong \left\{ \boldsymbol{p} \in \mathcal{K}_2 \;\middle|\; \begin{array}{c} \sum_{\mathbf{red}}(\boldsymbol{p}) = \sum_{\mathbf{red}}(\boldsymbol{q}) \\ \sum_{\mathbf{blue}}(\boldsymbol{p}) = \sum_{\mathbf{blue}}(\boldsymbol{q}) \end{array} \right\}$$

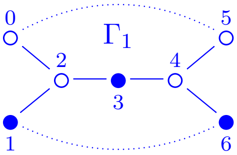
No groups, *two* congruences instead.

$H^1(\mathbb{R}, \mathrm{Spin}(12))$

For $\mathrm{Spin}(12)$ we have

$$\boldsymbol{q} = {}^{2}_{0}0\,0\,0^{0}_{0} \qquad \sum_{\text{red}}(\boldsymbol{q}) = 0, \quad \sum_{\text{blue}}(\boldsymbol{q}) = 0$$





We have 6 **even-even** labelings, hence $\#H^1(\mathbb{R}, \mathrm{Spin}(12)) = 6$:

$${}^{2}_{0}0\,0\,0^{0}_{0} \qquad {}^{0}_{0}0\,0\,0^{2}_{0} \qquad {}^{0}_{2}0\,0\,0^{0}_{0} \qquad {}^{0}_{0}0\,0\,0^{0}_{2}$$

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$H^1(\mathbb{R}, \mathrm{Spin}(12))$

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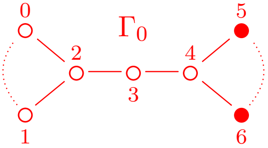

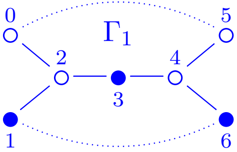

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$H^1(\mathbb{R}, \mathrm{Spin}(10,2))$

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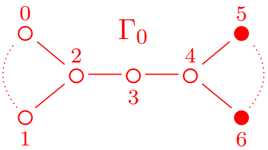

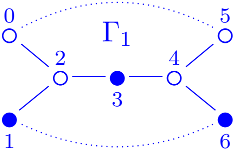

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For the quaternionic form: $\mathrm{Spin}(\boldsymbol{q}'') \simeq \mathrm{Spin}(\boldsymbol{q}') =: \mathrm{Spin}^*(12)$, and

$$\#H^1(\mathbb{R}, \mathrm{Spin}^*(12)) = 2.$$

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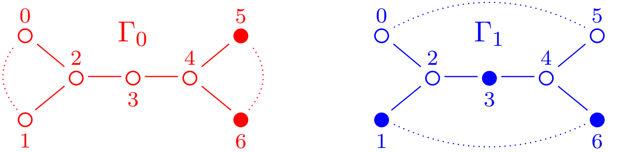

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The method of this talk works for all **semisimple** $\mathbb{R}$-groups, not necessarily adjoint or simply connected, at least when our group is an **inner** form of a compact group, i.e. has a compact maximal torus.

Note that all simple $\mathbb{R}$-groups of types $\mathbf{B}_\ell$, $\mathbf{C}_\ell$, $\mathbf{E}_7$, $\mathbf{E}_8$, $\mathbf{F}_4$ and $\mathbf{G}_2$ are **inner** forms of compact groups, because the corresponding Dynkin diagrams have no nontrivial automorphisms, hence these groups have no outer automorphisms.

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Cocycles

$$\text{Let } G = \ \mathrm{SO}(12),\ \mathrm{HSpin}(12),\ \mathrm{Spin}(12),\ \mathrm{PSO}(12).$$

I describe $\xi(\boldsymbol{p}) \in H^1(\mathbb{R}, G(\boldsymbol{q}))$, where $\boldsymbol{q}$ is a Kac 2-labeling, and $\boldsymbol{p}$ is a Kac 2-labeling *compatible with* $\boldsymbol{q}$. Let $T \subset G$ be a (compact) maximal torus, $\alpha_1, \dots, \alpha_6 \colon T_{\mathbb{C}} \to \mathbf{G}_{m,\mathbb{C}}$ the simple roots,

$$\mathfrak{t} = \mathrm{Lie}\, T, \qquad d\alpha_j \colon \mathfrak{t}_{\mathbb{C}} \to \mathbb{C}.$$

Define $x \in \mathfrak{t}$ and $c \in T(\mathbb{R})$ by

$$d\alpha_j(x) = i(p_j - q_j)/2, \quad \text{where } i^2 = -1,\ j = 1, 2, \dots, 6,$$
$$c = \exp 2\pi x \in T(\mathbb{R}).$$

Then $c \in T(\mathbb{R})_2 \subset Z^1(\mathbb{R}, G(\boldsymbol{q}))$, and we set

$$\xi(\boldsymbol{p}) = [c] \in H^1(\mathbb{R}, G(\boldsymbol{q})).$$

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Then $c \in T(\mathbb{R})_2 \subset Z^1(\mathbb{R}, G(\boldsymbol{q}))$, and we set

$$\xi(\boldsymbol{p}) = [c] \in H^1(\mathbb{R}, G(\boldsymbol{q})).$$

Cocycles

Let $G = \ \mathrm{SO}(12),\ \mathrm{HSpin}(12),\ \mathrm{Spin}(12),\ \mathrm{PSO}(12).$

I describe $\xi(\boldsymbol{p}) \in H^1(\mathbb{R}, G(\boldsymbol{q}))$, where $\boldsymbol{q}$ is a Kac 2-labeling, and $\boldsymbol{p}$ is a Kac 2-labeling *compatible with* $\boldsymbol{q}$. Let $T \subset G$ be a (compact) maximal torus, $\alpha_1, \ldots, \alpha_6 \colon T_{\mathbb{C}} \to \mathbf{G}_{m,\mathbb{C}}$ the simple roots,

$$\mathfrak{t} = \operatorname{Lie} T, \qquad d\alpha_j \colon \mathfrak{t}_{\mathbb{C}} \to \mathbb{C}.$$

Define $x \in \mathfrak{t}$ and $c \in T(\mathbb{R})$ by

$$d\alpha_j(x) = i(p_j - q_j)/2, \quad \text{where } i^2 = -1,\ j = 1, 2, \ldots, 6,$$
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Colorings

Q denotes the root lattice, P denotes the weight lattice,
$X = X^*(T_{\mathbb{C}})$ is the character lattice of $G = \mathrm{HSpin}(12)$, then

$$Q \subset X \subset P.$$

We have

$$Q = \langle \alpha_1, \alpha_2, \ldots, \alpha_6 \rangle, \quad X = \langle Q, \lambda \rangle$$

where $\lambda = (\alpha_1 + \alpha_3 + \alpha_6)/2$.

We color vertices 1,3, and 6 in blue.

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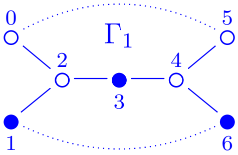
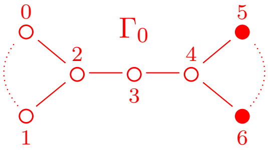


Similarly, for $G = \mathrm{SO}(12)$ we have

$$X = \langle Q, \lambda' \rangle, \quad \text{where } \lambda' = (\alpha_5 + \alpha_6)/2,$$

and we color vertices 5 and 6 in red.



The general formula for congruences

The formula that generalizes to all semisimple groups:

For any generator $[\lambda]$ of the finite abelian group X/Q, write $\lambda \in X$ as

$$\lambda = \sum_{j=1}^{6} c_j \alpha_j, \quad c_j \in \mathbb{Q},$$

then we require that

$$\sum_{j=1}^{6} c_j p_j \equiv \sum_{j=1}^{6} c_j q_j \mod \mathbb{Z}.$$

For $G = \mathrm{HSpin}(12)$ we have $X/Q \simeq \mathbb{Z}/2\mathbb{Z}$, and so we have one generator and hence, one congruence.

For $G = \mathrm{Spin}(12)$ we have $X/Q = P/Q \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and so we have two generators and hence, two congruences.

For $G = \mathrm{PSO}(12)$ we have $X/Q = Q/Q = 0$, no congruences.

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The general formula for the finite group

$P^\vee$ denotes the coweight lattice, it is dual to Q.
$Q^\vee$ denotes the coroot lattice, it is dual to P.
$X^\vee = X_*(T_{\mathbb{C}})$ is the cocharacter lattice of $G = \mathrm{HSpin}(12)$, it is dual to X, then

$$Q^\vee \subset X^\vee \subset P^\vee.$$

Then the finite abelian group $\Gamma = Z(\widetilde{G}) = P^\vee/Q^\vee$ acts on $\widetilde{D}$, and "our" finite group is $X^\vee/Q^\vee \subset \Gamma$.

For $G = \mathrm{HSpin}(12)$ we have

$$\Gamma_1 = X^\vee/Q^\vee \simeq \mathbb{Z}/2\mathbb{Z}.$$

For $G = \mathrm{Spin}(12)$ we have $X^\vee = Q^\vee$ (because the group is simply connected), hence $X^\vee/Q^\vee = 0$ - no group.

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Thank you!