The Use of Linear Algebraic Groups in Geometry and Number Theory

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1 Overview of the Field

The theory of linear algebraic groups is a well established area of modern mathematics. It started as an algebraic version of the massively successful and widely applied theory of Lie groups, pushed forward most notably by Chevalley and Borel. In the hands of Serre and Tits, it developed into a powerful tool for understanding algebra, geometry and number theory (e.g. Galois cohomology). In particular, it provides a way to unify seemingly distinct statements in algebra, geometry and number theory, hence, suggesting new techniques and methods for solving problems in these areas.

For example, the Hasse-Minkowski Theorem and the Albert-Brauer-Hasse-Noether Theorem, which concern respectively quadratic forms and central simple algebras over global fields, can be viewed as special cases of the celebrated Hasse Principle in Galois cohomology of semisimple linear algebraic groups (due to Kneser, Harder, and Chernousov) which unifies these two theorems and provides many new results that would not have been even suspected nor proven before.

This philosophy has led to a creation of a vast number of new techniques and applications to different areas of mathematics: the theory of quadratic forms (Karpenko, Merkurjev, Vishik), the theory of essential and canonical dimensions (Reichstein, Merkurjev), local-global principles (Hartmann, Harbater, Krashen, Parimala, Suresh), the theory of motives (Petrov, Semenov, Zainoulline), and to the theory of torsors (Chernousov, Gille, Panin, Pianzola). For instance, Karpenko has recently shown how results on isotropy of hermitian forms (linear algebraic groups of type A) can imply corresponding results on symplectic forms and quadratic forms (groups of types B, C, D), and how the results are all connected via the theory of algebraic cycles and motives of projective homogeneous varieties.

Further afield are applications to arithmetic groups and arithmetic locally symmetric spaces (Prasad, Rapinchuk) and to number theory in the form of pseudo-reductive groups (Conrad). In the opposite direction, another trend has been using results from finite group theory to prove theorems about algebraic groups (Guralnick).

The purpose of the workshop was to exploit and to develop these new emerging links, to bring together specialists and young researchers from these areas, to stimulate new advances and developments. More precisely, it was devoted to recent applications of linear algebraic groups in algebra, geometry and number theory, especially, to techniques and results that establish new links between different areas of mathematics, such as: the proof of the Grothendieck-Serre conjecture; the breakthrough in the computation of cohomological invariants based on new Merkurjev's results concerning motivic cohomology; Conrad's proof of finiteness of fibers for fppf cohomology over global fields of prime characteristic; and applications of representations with dense orbits inspired by Bhargava's work.

2 Recent Developments and Open Problems

The last five years can be characterized as a boom of research activity in the area of linear algebraic groups and its applications. One should mention here recent results

On the proof of the Grothendieck-Serre conjecture. We recall that the geometric case of the Grothendieck-Serre conjecture stated in the mid 60's says that if a G-torsor (defined over a smooth algebraic variety X) is rationally trivial, then it is locally trivial (in the Zariski topology), where G is a smooth reductive group scheme over X. This conjecture has a long history: It was proven for curves and surfaces for quasi-split groups by Nisnevich in the mid-1980's. For arbitrary tori in the late 1980's by Colliot-Thelene and Sansuc. If G is defined over a field, then the conjecture is known as Serre's conjecture and was proven by Colliot-Thelene, Ojanguren and Raghunathan in the beginning of the 1990's; for most classical groups in the late 1990's by Ojanguren-Panin-Suslin-Zainoulline; for isotropic groups by Panin-Stavrova-Vavilov. However, no general proof was known up to now. Recently, Fedorov and Panin found a new original approach that proves the conjecture using the theory of affine Grassmannians coming from Langlands' program assuming the base field is infinite. Finally, Panin extended that approach to include the case of an arbitrary field.

On computations of the group of cohomological invariants. According to J.-P. Serre by a cohomological invariant one means a natural transformation from the first Galois cohomology with coefficients in an algebraic group G (the pointed set which describes all G-torsors) to a cohomology functor $h(-)$, where h is a Galois cohomology with torsion coefficients, a Witt group, a Chow group with coefficients in a Rost cyclic module M, etc. The ideal result here would be to construct enough invariants to classify all G-torsors/linear algebraic groups. The question was put on a firm foundation by Serre and Rost in the 1990's, allowing the proof of statements like "the collection of cohomological invariants of G is a free module over the following cohomology ring" for certain groups G; this theory is expounded in the 2003 book by Garibaldi-Merkurjev-Serre. Using this theory one obtains complete description of all cohomological invariants landing in degree 1 Galois cohomology for all algebraic groups, in degree 2 for connected groups, and in degree 3 for simply connected semisimple groups (Rost). In a breakthrough recent development, Merkurjev provided a complete description of degree 3 invariants for semisimple groups, solving a long standing question. This has ignited new activity, with researchers trying to understand the full power of his new methods, as well as to understand the new invariants discovered as a corollary of his results.

On length spectra of locally symmetric spaces. The answer to the question "Can you hear the shape of a drum?" is famously no, but variations of the problem such as restricting the collection of spaces under consideration or strengthening the hypotheses has led to situations where the answer is yes. In a remarkable 2009 Publ. Math. IHES paper, Prasad and Rapinchuk introduced the notion of weak commensurability of semisimple elements of algebraic groups and of arithmetic groups and used this new concept to address the question of when arithmetically defined locally symmetric spaces have the same length spectrum. In this paper they also settled many cases of the long-standing question of when algebraic groups with the same maximal tori are necessarily isomorphic. This paper and the stream of research stemming from it connects algebraic groups and their Galois cohomology – the central subject of this conference – with arithmetic groups, with geometry, and even with transcendental number theory.

On applications of the algebraic cycles and Grothendieck gamma filtration to the invariants of torsors. Let X be the variety of Borel subgroups of a simple linear algebraic group G over a field k. It was proven that the torsion part of the second quotient of Grothendieck's gamma-filtration on X is closely related to the torsion of the Chow group and hence to the group of cohomological invariants in degree 3 computed recently by Merkurjev. As a byproduct of this new striking connection one obtains an explicit geometric interpretation/description of various cohomological invariants in degree 3 as well as obtains new results concerning algebraic cycles and motives of projective homogeneous spaces (Baek-Garibaldi-Gille-Junkins-Queguiner-Semenov-Zainoulline).

On genericity theorems for the essential dimension of algebraic stacks and their applications. Techniques from the theory of algebraic stacks are used to prove genericity theorems (Brosnan-Reichstein-Vistoli) to bound their essential dimension which is then applied to finding new bounds for the more classical essential dimension problems for algebraic groups, forms and hypersurfaces. These genericity theorems have also been used in particular by Biswas, Dhillon and Lemire to find bounds on the essential dimension of stacks of (parabolic) vector bundles over curves and by Biswas, Dhillon, Hoffman to find bounds on the essential dimension of the stack of coherent sheaves over a curve.

On the classification of simple stably Cayley groups. A linear algebraic group is called a Cayley group if it is equivariantly birationally isomorphic to its Lie algebra. It is stably Cayley if the product of the group and some torus is Cayley. Cayley gave the first examples of Cayley groups with his Cayley map back in 1846. Over an algebraically closed field of characteristic 0, Cayley and stably Cayley simple groups were classified by Lemire-Popov-Reichstein in 2006. In 2012, the classification of stably Cayley simple groups was extended to arbitrary fields of characteristic 0 by Borovoi-Kunyavskii-Lemire-Reichstein. Borovoi-Kunyavskii then used this classification to classify the stably Cayley semisimple groups over arbitrary fields of characteristic 0.

On classifications of finite groups of low essential dimension. Duncan used the classification of minimal models of rational G-surfaces in his classification of the finite groups of essential dimension 2 over an algebraically closed field of characteristic 0. Beauville more recently used Prokhorov's classification of rationally connected threefolds with an action of a simple group to classify the finite simple groups of essential dimension 3.

Patching techniques. There are open questions concerning Hasse principle for homogeneous spaces under linear algebraic groups over function fields of p-adic curves. There is a surge of activities in this direction, thanks to the patching techniques and theorems due to Harbater-Hartmann-Krashen. This has led to answers to questions on Hasse principle for homogeneous spaces over several classes of linear algebraic groups. There are also answers to fundamental questions related to period-index questions for the Brauer group u-invariant of fields. New conjectures were formulated by Colliot-Thélène, Parimala and Suresh very similar to the conjectures over number fields. Reciprocity obstructions similar to the Brauer-Manin obstruction were constructed to study the obstruction to the Hasse principle. There are a host of open questions concerning the Hasse principle for rational groups which makes this area truly challenging.

3 Presentation Highlights

I. The first day of the workshop was devoted to general talks on the structure of linear algebraic groups and their cohomological invariants. There were morning talks by senior researcher A. Merkurjev (UCLA, USA) and 2 young researchers S. Baek (KAIST, South Korea) and R. Pirisi (Ottawa). The afternoon talks were given by senior researchers V. Chernousov (Alberta), M. Borovoi (Tel Aviv, Israel) and young researchers I. Rapinchuk (Harvard), D. Zywina (Cornell, USA).

Speaker: **Alexander Merkurjev** (University of California at Los-Angeles)
Title: *Suslin's Conjecture on the reduced Whitehead group of a simple algebra*
In the talk, the speaker reported about his proof of Suslin's Conjecture (1991) on the generic non-triviality of the reduced Whitehead group of a simple algebra. For a central simple algebra A over a field F, the reduced Whitehead group for $G = SL_1(A)$ is $\mathrm{SK}_1(A) = G(F)/[A^\times, A^\times]$. Saltman showed that G is retract rational if and only if $\mathrm{SK}_1(A_K) = 0$ for all field extensions K/F. Wang in 1950, showed that if the index of A is squarefree, then $\mathrm{SK}_1(A) = 0$. However, Platonov in 1975 gave an example of a central simple algebra A of index p^2 with non-trivial reduced Whitehead group. This and other counterexamples

led to Suslin's Conjecture in 1991 that $\mathrm{SK}_1(A)$ is non-trivial generically when $\mathrm{ind}(A)$ is not squarefree. In 1993, Merkurjev proved the conjecture in the case when 4 divides the index and when $\mathrm{char}(F) \neq 2$. In 2006, Merkurjev reduced the conjecture in a characteristic free way to statements about the Chow ring of G. Namely, it is necessary to show that if ind$(A) = p$, then for all field extensions K/F, $\mathrm{CH}^*(G) \to \mathrm{CH}^*(G_K)$ is surjective and if $\mathrm{ind}(A) = p^2$, then $\mathrm{CH}_i(\mathrm{SB}(A))$ are torsion free for all $i = 0, \dots, p-2$. Recently, he proved these equivalent statements using information about the topological filtration for Severi-Brauer varieties, the decomposition of the Chow motive for a Severi-Brauer variety and spectral sequence arguments. [13]

Speaker: **Sanghoon Baek** (Korean Advanced Institute of Science and Technology)
Title: *Semi-decomposable invariants of degree 3*
The speaker reported on semi-decomposable invariants in degree 3 for split semisimple groupss, introduced by Merkurjev-Neshitov-Zainoulline. This invariant is locally decomposable and it was shown that there is no nontrivial semi-decomposable invariant of a split simple group. In this talk, he discussed semi-decomposable invariants of a split reductive group in terms of the torsion in the codimension 2 Chow groups of a product of Severi-Brauer varieties. In particular, he presented a method to find nontrivial semi-decomposable invariants of an arbitrary split semisimple group of type A. [2]

Speaker: **Vladimir Chernousov** (University of Alberta)
Title: *Algebraic groups and their maximal tori*
The speaker surveyed recent developments dealing with characterization of absolutely almost simple algebraic groups having the same isomorphism/isogeny classes of maximal tori over the field of definition. These questions arose in the analysis of weakly commensurable Zariski-dense subgroups. In particular, for an algebraic K group, he discussed $\mathrm{gen}_K(G)$, the number of isomorphism classes of K forms of G with the same maximal tori. In the case of a finitely generated field K, he asked when $\mathrm{gen}_K(G) = 1$ or, at least finite. His talk discussed the case of absolutely almost simple K groups over number fields. Then in joint work with A. Rapinchuk and I. Rapinchuk, they showed that $\mathrm{gen}_K(G)$ was finite in that case and in fact was reduced to one element except for types A_n, D_{2n+1} or E_6. He then discussed their conjecture that $\mathrm{gen}_K(G)$ is finite over a finitely generated field of characteristic 0 or of good characteristic. He gave supporting evidence for their conjecture. [6]

Speaker: **Roberto Pirisi** (University of Ottawa)
Title: *Cohomological Invariants for stacks of algebraic curves*
The speaker discussed how to extend the classical theory to a theory of cohomological invariants for Deligne-Mumford stacks and in particular for the stacks of smooth genus g curves. He also showed how to compute the additive structure of the ring of cohomological invariants for the algebraic stacks of hyperelliptic curves of all even genera and genus three. [16]

Speaker: **Mikhail Borovoi** (Tel Aviv University)
Title: *Real Galois cohomology of semisimple groups*
In joint work with D. Timashev, the speaker explained how to compute the Galois cohomology set $H^1(\mathbb{R}, G)$ for a connected semisimple algebraic real group G using Kac diagrams, introduced by Kac to describe automorphisms of finite order of simple Lie algebras over the field of complex numbers $\mathbb{C}$. [4]

Speaker: **David Zywina** (Cornell University)
Title: *l-adic monodromy groups for abelian varieties*
For an abelian variety of dimension $g \geq 1$ over a number field K and a prime l, the l^m torsion subgroup of $A(\overline{K})$ is isomorphic to $(\mathbb{Z}/l^m\mathbb{Z})^{2g}$. This means that for each prime L, one obtains a representation of Gal_K in $\mathrm{GL}_{2g}(\mathbb{Z}/l^m\mathbb{Z})$. Combining these representations, one obtains a representation $\rho_{A,l^\infty} : \mathrm{Gal}_K \to \mathrm{GL}_{2g}(\mathbb{Z}_l)$. After increasing the size of K, $\rho_{A,l^\infty}(\mathrm{Gal}_K)$ is a finite index subgroup of $G_l(\mathbb{Q}_l)$ where G_l is the Zariski closure of the abelian variety in $\mathrm{GL}_{2g}/\mathbb{Q}_l$. The Mumford Tate conjecture states that G_l is isomorphic to $G \times_{\mathbb{Q}} \mathbb{Q}_l$ where G is the Mumford Tate group constructed from the Hodge decomposition of $A/\mathbb{C}$.

The speaker discussed the generic factorisation of the reduction of an abelian variety over a number field K with respect to a prime of good reduction of the algebraic integers of K assuming the Mumford Tate conjecture.

II. The second day of the conference featured talks about cohomological invariants and patching techniques for function fields over p-adic curves, consequences of the proof of the Grothendieck-Serre conjecture, results on essential dimension and rationality, classification of simple isotropic groups and results on quadratic forms in characteristic 2. There were 1 hour talks by E. Bayer-Fluckiger (EPFL, Switzerland), V. Suresh (Emory, USA), J. Hartmann (Aachen, Germany) and half-hour talks by young researchers A. Auel (Yale, USA), M. Macdonald (Lancaster, UK), A. Stavrova (St Petersburg, Russia), A. Duncan (South Carolina, USA) and doctoral student N. Bhaskhar (Emory, USA).

Speaker: **Eva Bayer-Fluckiger** (École Polytechnique Fédérale de Lausanne)
Title: *Rationally isomorphic hermitian forms and torsors of some non-reductive groups*
A well known theorem about unimodular quadratic forms over a discrete valuation ring R and its quotient field F says that if two unimodular quadratic forms over R are isomorphic over F, then they become isomorphic over R. This result and many of its generalisations are consequences of the Grothendieck Serre conjecture which states that for a regular local integral domain with fraction field F. For any smooth reductive affine group scheme G over R, the induced map $H^1_{ét}(R, G) \to H^1_{ét}(F, G)$ is injective. Grothendieck-Serre was proved recently in the case in which R contains a field by Panin-Fedorov. One can also make this conjecture for non-connected group schemes whose connected components is reductive. The orthogonal group and its forms is a particular case of interest. Let (P, f) be a unimodular quadratic space and $\mathbf{O}(f)$ is a group scheme of isometries of f. Then the first theorem mentioned is a consequence of the analogue of Grothendieck-Serre for $\mathbf{O}(f)$ and was proved in the case of dimension of R at most 2 or when R contains a field k.

In a different direction, the first theorem was recently generalised by Auel, Parimala and Suresh to show that two quadratic forms on a semilocal Dedekin domain R containing 2 in the units, that if f, f' are quadratic forms over R with isomorphic simple coradicals, then if they are isomorphic over F, they are isomorphic over R. This result no longer follows from Grothendieck Serre since the forms are not necessarily unimodular.

The speaker discussed joint work with Uriya First dealing with generalisations of the result of Auel, Parimala and Suresh and hence generalisations of the Grothendieck-Serre conjecture in the non-reductive case. For a semilocal Dedekind domain R with $2 \in R^\times$ and its quotient field F, they proved that for two unimodular hermitian forms over (A, σ) for a hereditary R-order in a separable F-algebra and σ an R involution, then if the associated unimodular hermitian forms are isomorphic over F, then they are isomorphic over A. In this setting, this result has a cohomological analogue which states the map $H^1_{ét}(R, \mathbf{O}(f)) \to H^1_{ét}(F, \mathbf{O}(f))$ is injective where $\mathbf{O}(f)$ is the group scheme of isometries of f. Note here that $\mathbf{O}(f)$ has a connected component which is not reductive so that the usual Grothendieck-Serre conjecture does not apply. The result can be regarded as a first step towards a version of the Grothendieck-Serre conjecture for certain non-reductive group schemes over $\mathrm{Spec}(R)$.

She also talked about an equivariant version of that theorem about hermitian forms invariant under the action of a finite group. [3]

Speaker: **Asher Auel** (Yale University)
Title: *Algebras of composite degree split by genus one curves*
The speaker addressed the old question of whether every central simple algebra can be split by the function field of a genus one curve defined over the base field. The speaker outlined the ideas of the proofs for the known cases of algebras of degree at most 5 due to work of Artin, Swets, Clark and de Jong-Ho. He proposed a method for answering this question in the affirmative for algebras of composite degree, when the answer is known for the prime power factors. The speaker illustated his method for degree 6 and outlined the proof in that case.

Speaker: **Mark MacDonald** (University of Lancaster)
Title: *Reducing E_7 and the slice method*
The speaker discussed joint work with R. Loetscher in which they gave a definition of a (G, N) slice for an algebraic group G over a field k, X a G scheme and $N \leq G$ a subgroup, generalizing definitions by Seshadri from the sixties and Katsylo from the eighties. Their definition of a (G, N) slice is a (locally closed) N-stable subscheme $S \subset X$ such that the induced morphism $(G \times S)/N \to X$ of algebraic spaces is an open immersion. Here N acts freely on $G \times S$ via $n(g, s) = (gn^{-1}, ns)$. One of the motivations for defining (G, N) slices $S \subseteq X$ is that $k(X)^G \cong k(S)^N$ which has been used to good effect to simplify the rationality

problem. They showed that the existence of a (G, N) slice of a versal G-scheme implies the surjectivity of the maps $H^1_{fppf}(L, N) \to H^1_{fppf}(L, G)$ for infinite field L containing F. Such a result implies that $\mathrm{ed}(G) \le \mathrm{ed}(N)$ where ed refers to the essential dimension of the algebraic group. This result generalises a result of Reichstein in the characteristic zero case for linear representations of G. The speaker discussed a construction of slices of geometrically irreducible G-varieties coming from stabilizers in general position (SGPs). He showed that for a SGP for a geometrically irreducible G-variety V, there exists a $(G, N_G(H))$ slice which is open in the fixed scheme V^H. Combining this result with their result on essential dimension and slices, they were able to get a new bound on the essential dimension of the split simply connected group of type E_7. [12]

Speaker: **Venapally Suresh** (Emory University)
Title: *Rost invariant over function fields of p-adic curves*
Let F be a field and G an absolutely almost simple simply connected algebraic group over F. For the Rost invariant $H^1(F, G) \to H^3(F, \mathbf{Q}/\mathbf{Z}(2))$, the speaker discussed the injectivity of this map when F is the function field of a p-adic curve, with special reference to $G = SL_1(A)$, where A is a central simple algebra over F of index coprime to p.

Speaker: **Nivedita Bhaskhar** (Emory University)
Title: *Reduced Whitehead groups of division algebras over function fields of p-adic curves*
The question of whether every reduced norm one element of a central simple algebra is a product of commutators was formulated in 1943 by Tannaka and Artin in terms of the reduced Whitehead group $SK_1(D)$.

The speaker addressed the question of the triviality of the reduced Whitehead group for for l torsion, degree l^2 algebras over function fields of p-adic curves where l is any prime not equal to p. The proof relies on the techniques of patching as developed by Harbater-Hartmann-Krashen and exploits the arithmetic of these fields to show triviality of the reduced Whitehead group.

Speaker: **Julia Hartmann** (RWTH Aachen)
Title: *Obstructions to Local-Global Principles for Linear Algebraic Groups*
Local-Global Principles are a very important subject in the theory of linear algebraic groups. The speaker discussed such principles for groups defined over arithmetic function fields. In that case, some obstructions come from a collection of overfields associated with patching. These obstructions are reasonably well understood for rational linear algebraic groups, but interesting examples due to Colliot-Thélène, Parimala and Suresh show that the case of nonrational groups is more difficult. Recent joint results with D. Harbater and D. Krashen provide an explanation as to why these examples occur, via the geometry of a model for the function field. [7]

Speaker: **Anastasia Stavrova** (St.Petersburg University)
Title: *Simple algebraic groups and structurable algebras*
The speaker presented a uniform proof of the well-known correspondence between isotropic simple algebraic groups and simple structurable or Jordan algebras in joint work with T. De Medts and L. Boelaert.

Speaker: **Alexander Duncan** (University of South Carolina)
Title: *Pairs of quadratic forms in characteristic 2*
In joint work with I. Dolgachev, the speaker considered smooth complete intersections of two quadrics in even-dimensional projective space. Over an algebraically closed field of characteristic not 2, it is well known that one can find a basis in which both quadratic forms are diagonal. However, this fails in characteristic 2. He presented a normal form which applies over an arbitrary field of characteristic 2. The normal form can be used to determine the automorphism groups of these varieties.

III. The third day of talks featured talks on realizing algebraic groups as automorphism groups, invariant theory for automorphism groups of simple algebras, equivariant oriented cohomology and motives of twisted flag varieties, rational orbits for groups acting on varieties, and period-index problems. One hour talks were given by M. Brion (Institut Fourier, France), V.L. Popov (Steklov Institute, Russia), G. Savin (Utah, USA).

Half hour talks were given by young researchers C. Zhong (SUNY Albany,USA), O. Haution (Munich, Germany), B. Antieau (UIC, USA) and doctoral student A. Neshitov (Ottawa/Steklov).

Speaker: **Michel Brion** (Institut Fourier)
Title: *Realizing algebraic groups as automorphism groups*
The speaker addressed the question of realizing a given algebraic group as the automorphism group of some algebraic variety. He showed that every smooth connected group scheme over a perfect field is the connected automorphism group scheme of a normal projective variety. In the characteristic zero case, one could take the variety to be smooth. For a finite dimensional Lie algebra over a field of characteristic zero, he gave equivalent conditions for the Lie algebra to be derivations of the structure sheaf of some proper scheme over the field. [5]

Speaker: **Changlong Zhong** (State University of New-York at Albany)
Title: *Equivariant oriented cohomology of flag varieties*
In joint work with Calmès and Zainouilline, the speaker explained an algebraic construction of equivariant oriented cohomology of (partial and full) flag varieties and of the push-pull morphisms between these cohomology groups. In particular, he showed how for an equivariant oriented cohomology theory h over a base field k, a split reductive group G over k, a maximal torus T in G and a parabolic subgroup P containing T, the equivariant oriented cohomology ring $h_T(G/P)$ can be associated with a formal affine Demazure algebra which is the dual of a coalgebra and can be defined just using the root datum of (G, T), a set of simple roots defining P and the formal group law of h. With respect to these algebras, he showed how operators can be defined to construct push-forwards and pull-backs along geometric morphisms. A Schubert Calculus was described for these rings. [19]

Speaker: **Olivier Haution** (University of Munich)
Title: *Finite group actions on the affine space*
The speaker discussed the existence of fixed rational points for the action of a finite p-group on affine n space over a field of characteristic different than p. This question was popularised in a paper of Serre from 2009 who proved this in a number of important cases, and pointed out that the answer was unknown when the group is cyclic of order 2, the field is $\mathbb{Q}$ and $n = 3$. The list of positive known cases was extended by Esnault and Nicaise in 2011. The speaker proved the existence of a rational fixed point when k is l-special for some prime different from its characteristic and when k is perfect and fertile and $n = 3$. [8]

Speaker: **Alexander Neshitov** (University of Ottawa / Steklov Institute at St.Petersburg)
Title: *Motives of twisted flag varieties and representations of Hecke-type algebras*
In joint work in progress with N. Semenov, V. Petrov and K. Zainoulline, the speaker related the category of (cobordism-) Ω-motives of twisted flag varieties for a semisimple linear algebraic group G with the category of integer (or modular) representations of the associated Hecke-type algebra $H = H(G)$.

The algebra H was introduced and studied recently in a series of papers by Calmès, Hoffnung, Malagon-Lopez, Savage, Zainoulline, Zhao, Zhong. It has two important properties: (i) its dual over $\Omega_T(pt)$, where T is a split maximal torus of G, gives the T-equivariant cobordism ring $\Omega_T(G/B)$ of the variety of Borel subgroups of G; (ii) its complete set of generators and relations is known and resembles those of an affine Hecke algebra. [14]

Speaker: **Vladimir L. Popov** (Steklov Institute, Moscow)
Title: *Simple algebras and algebraic groups*
The speaker discussed the following questions:

(1) Given an algebraic group G, let V be a finite-dimensional algebraic G-module that admits a structure of a simple (not necessarily associative) algebra A for which $G = \mathrm{Aut}(A)$. Then V admits a close approximation to the analogue of classical invariant theory.

(2) What are the groups G for which such a V exists?

(3) Given G, what are the G-modules V for which (1) holds?

Speaker: **Gordan Savin** (Utah University)
Title: *Twisted Bhargava Cubes*
In joint work with Wee Teck Gan, the speaker discussed the problem of classifying rational orbits for pre homogeneous spaces. A classical example is $GL(n)$ acting on the space of symmetric matrices. In this case rational orbits are parameterized by isomorphism classes of quadratic spaces. For some pre homogeneous spaces arising from exceptional groups the orbit problem has an answer in terms of (twisted) composition algebras. [18]

Speaker: **Benjamin Antieau** (University of Illinois at Chicago)
Title: *Prime decomposition in period-index problems via representation theory*
The speaker reported on joint work with B. Williams on the use of representations of projective general linear groups to extend known facts about the prime divisors of the period and index of Brauer classes that hold over fields to more general settings. The first result is that the primes dividing the period and index agree. This is proved using only exterior representations. The second result is that the index of a Brauer class is the product of the indices of each of its p-parts. This requires more complicated Young diagrams. [1]

IV. The fourth day featured morning talks about oriented motivic theories by Mark Levine (Essen, Germany), generalisations of the Grothedieck Serre conjecture by I. Panin (Steklov Institute, Russia) and the topological index of Brauer classes by young researcher B. Williams (UBC). There was a free afternoon.

Speaker: **Marc Levine** (University of Essen-Duisburg)
Title: *On the geometric part of some oriented motivic theories*
For an oriented motivic ring spectrum E in $SH(k)$, k a field of characteristic zero, there is a canonical map

$$\Omega^* \to E^{2*,*}$$

of oriented cohomology theories on Sm/k, in the sense of Levine-Morel. If E has associated formal group law (F, R), this map descends to

$$\Omega^* \otimes_L R \to E^{2*,*}$$

In joint work with S. Dai and G. Tripathi, the speaker described a criterion which implies that this second map is an isomorphism of oriented cohomology theories on Sm/k. He showed that for a wide class of examples, including MGL, Landweber exact theories and their connective covers as well as certain quotients or localizations of MGL, such as truncated Brown-Peterson theories, Morava K-theories and connective Morava K-theory, satisfy this criterion. [11]

Speaker: **Ivan Panin** (Steklov Institute at St.Petersburg)
Title: *A purity theorem*
The speaker discussed the following:

Conjecture. Let $\mathcal{O}$ be a regular local ring and K be its fraction field. Let $m\colon G \to C$ be a smooth $\mathcal{O}$-morphism of reductive $\mathcal{O}$-group schemes, with a torus C. Suppose additionally that the kernel of m is a reductive $\mathcal{O}$-group scheme. Then the following sequence

$$\{1\} \to C(\mathcal{O})/m(G(\mathcal{O})) \to C(K)/m(G(K)) \to \bigoplus_{ht(p)=1} C(K)/[C(\mathcal{O}_p) \cdot m(G(K))] \to \{1\}$$

is exact, where p runs over all height 1 primes of $\mathcal{O}$ and

$$res_p\colon C(K)/m(G(K)) \to C(K)/[C(\mathcal{O}_p) \cdot m(G(K))]$$

is the natural map (the projection to the factor group).

Theorem. The conjecture is true, if $\mathcal{O}$ is a regular local ring containing a field.

Remark. The exactness of that sequence in the middle term is used in the proof of the Grothendieck–Serre conjecture for regular local rings containing a field. [15]

Speaker: **Ben Williams** (University of British Columbia)
Title: *The topological index of period-2 Brauer classes*
The speaker outlined how one can use the homotopy theory of classifying spaces of linear groups to find obstructions to representing Brauer classes as the classes of Azumaya algebras of specific ranks, concentrating on the case of period-2 classes.

V. The last day featured talks by N. Karpenko (Alberta) about 16-dimensional quadratic forms and D. Krashen (UGA, USA) about the Clifford algebra of a morphism.

Speaker: **Nikita Karpenko** (University of Alberta)
Title: *On* 16*-dimensional quadratic forms in* I^3
The speaker discussed the mysteries related to quadratic forms with Witt class in I^3 focussing on the following questions for which 16 is the smallest dimension in which they were not understood:

- whether the forms can be can be parameterized by algebraically independent variables,
- if every form contains a proper subform with Witt class in I^2,
- how many 3-Pfister forms are needed to write the Witt class of an arbitrary 16-dimensional form in I^3 (over an arbitrary field) as their linear combination (no upper bound at all is available). [9]

Speaker: **Danny Krashen** (UGA)
Title: *The Clifford algebra of a finite morphism of schemes*
In joint work with M. Lieblich, the speaker defined a Clifford algebra associated to a finite morphism of schemes, generalizing the notion of the Clifford algebra of a homogeneous polynomial. He defined a Clifford functor from the category of finite surjective morphisms of proper k schemes to k sheaves and showed it was representable. He showed that the stack of representations of rank n of the Clifford algebra of a finite morphism of degree d is equivalent to the stack of Ulrich bundles of degree m over the finite morphism. He then described connections with relative Brauer groups and index reduction, with Ulrich bundles, and with the period-index problem for genus 1 curves. [10]

4 Scientific Progress Made

During the meeting, new and exciting results were reported on in the following areas:

- Merkurjev's remarkable proof of Suslin's conjecture on the generic non-triviality of the reduced Whitehead group.
- Panin's work on the Grothendieck Serre conjecture in the case of regular local rings containing a finite field.
- Work of Calmès, Neshitov, Zainoulline, Zhong to give an algebraic description of the equivariant oriented cohomology theory of flag varieties.
- Work of Neshitov, Semenov, Petrov and Zainouilline connecting motives of twisted flag varieties to representations of Hecke-type algebras.
- Krashen and Lieblich's description of the Clifford algebra of a finite morphism of schemes and applications of their methods to period index problems for curves of genus 1 and the theory of Ulrich bundles.
- Antieau and Williams' use of topological and representation theoretic methods to address period-index problems in general settings.

5 Outcome of the Meeting

The workshop attracted 42 leading experts and young researchers from Canada, France, South Korea, Germany, Israel, Russia, Switzerland, USA. There were 27 speakers in total: 13 talks were given by senior speakers, 12 talks by young researchers and postdocs and 2 talks by doctoral students.

The lectures given by senior speakers provided an excellent overview on the current state of research in the theory of algebraic groups in geometry and number theory. There were several new results announced, e.g. Merkurjev (on triviality of reduced Whitehead group), Panin (on Grothendieck-Serre conjecture), Krashen (on Clifford algebras and Ulrich bundles). The afternoon sessions provided a unique opportunity for young speakers to present their achievements. Numerous discussions between the participants after the talks have already led to several joint projects, e.g. by Junkins-Krashen-Lemire, Calmès-Neshitov-Zainoulline, Karpenko-Merkurjev.

The organizers consider the workshop to be a great success. The quantity and quality of the students, young researchers and the speakers was exceptional. The enthusiasm of the participants was evidenced by the frequent occurrence of a long line of participants waiting to ask questions to the speakers after each lecture. The organizers feel that the material these participants learned during their time in BIRS will prove to be very valuable in their research and will undoubtedly have a positive impact on the research activity in the area.

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