The Covering Index of Convex Bodies

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Covering by homothets and illumination

Let $\mathbb{E}^d$ denote the d-dimensional Euclidean space. A *d-dimensional convex body* K is a compact convex subset of $\mathbb{E}^d$ with nonempty interior.
A *homothet* of K is a set of the form $\lambda K + x$, where λ is a nonzero real number and $x \in \mathbb{E}^d$.

Conjecture 1 (Covering Conjecture)

K can be covered by 2^d of its smaller positive homothets and 2^d homothets are needed only if K is an affine d-cube.

The *illumination number* $I(K)$ of K is the smallest n for which the boundary of K can be illuminated by n points/directions.

Boltyanski (1960) showed that $I(K) = n$ if and only if the smallest number of smaller positive homothets of K that cover K is n.

Conjecture 2 (Illumination Conjecture)

$I(K) \leq 2^d$, and $I(K) = 2^d$ only if K is an affine d-cube.

The illumination parameter

Question: *How 'economically' can we illuminate K by a few nearby point sources?*

Bezdek (1992) introduced the *illumination parameter* $\mathrm{ill}(K)$ of a d-dimensional o-symmetric convex body K.

$$\mathrm{ill}(K) = \inf \left\{ \sum_i \|p_i\|_K : \{p_i\} \text{ illuminates } K, p_i \in \mathbb{E}^d \right\},$$

where $\|x\|_K = \inf\{\lambda > 0 : x \in \lambda K\}$ is the Minkowski functional of the symmetric convex body K.

- Far away light sources are penalized.
- $I(K) \leq \mathrm{ill}(K)$, for o-symmetric convex bodies.
- Bezdek (1992, 2006), Bezdek, Boroczky and Kiss (2006), Kiss and De Wet (2010) and others calculated $\mathrm{ill}(K)$ for various symmetric convex bodies.

The covering parameter

Question: *How 'economically' can we cover K by a few small homothets?*

Swanepoel (2005) defined the *covering parameter* of a d-dimensional convex body.

$$C(K) = \inf\left\{\sum_i \frac{1}{1-\lambda_i} : K \subseteq \bigcup_i (\lambda_i K + t_i), 0 < \lambda_i < 1, t_i \in \mathbb{E}^d\right\}.$$

- Large homothets are penalized.
- $\mathrm{ill}(K) \leq 2C(K)$, if K is o-symmetric.
- $C(K) = O(2^d d^2 \ln d)$, if K is o-symmetric.
- $C(K) = O(4^d d^{3/2} \ln d)$, in general.
- Let C^d denote a d-dimensional cube, then $C(C^d) = 2^{d+1}$.

Two ingredients: the Banach-Mazur distance

The (multiplicative) Banach-Mazur distance between d-dimensional convex bodies K and L is

$$d_{BM}(K, L) = \inf \{\delta \geq 1 : a \in K, b \in L, L - b \subseteq T(K - a) \subseteq \delta(L - b)\},$$

where the infimum is taken over all invertible linear operators $T : \mathbb{E}^d \longrightarrow \mathbb{E}^d$.

Denote by $\mathcal{K}^d$ the (compact) space of d-dimensional convex bodies under Banach-Mazur distance.

- d_{BM} is a multiplicative metric on $\mathcal{K}^d$.
- Used to study affine invariant functionals of convex bodies.
- Can think of d_{BM} as distance between d-dimensional real Banach spaces.

Two ingredients: $\gamma_m(K)$

Define $\gamma_m(K)$ to be the minimal homothety ratio required to cover K by m positive homothets.

$$\gamma_m(K) = \inf\left\{\lambda > 0 : K \subseteq \bigcup_{i=1}^{m}(\lambda K + t_i), t_i \in \mathbb{E}^d, i = 1, \ldots, m\right\}.$$

- Originally, introduced by Lassak (1986).
- Zong (2010) reintroduced it as a functional on $\mathcal{K}^d$ and proved it to be uniformly continuous.
- We observed that in fact, $\gamma_m(K) \leq d_{BM}(K, L)\gamma_m(L)$, for any $K, L \in \mathcal{K}^d$.

The covering index

Let $\mathcal{K}^d$ denote the space of d-dimensional convex bodies.
We define the *covering index* of K as

$$\mathrm{coin}(K) = \inf\left\{\frac{m}{1-\gamma_m(K)} : \gamma_m(K) \le 1/2, m \in \mathbb{N}\right\}. \qquad (1)$$

Intuitively, $\mathrm{coin}(K)$ measures how well K can be covered by a relatively small number of positive homothets all corresponding to the same relatively small homothety ratio.

Why $\gamma_m(K) \le 1/2$?

1) Rogers (1963), Verger-Gaugry (2005), O'Rourke (2012) and others investigated the minimum number of homothets of ratio $1/2$ or less needed to cover a d-dimensional ball.

2) Easy to obtain optimizers.

3) Define

$$f_m(K) = \begin{cases} \dfrac{m}{1-\gamma_m(K)}, & \text{if } \gamma_m(K) \le 1/2, \\ \infty, & \text{otherwise.} \end{cases}$$

Then $\mathrm{coin}(K) = \inf \{f_m(K) : m \in \mathbb{N}\}$.

For any $K, L \in \mathcal{K}^d$ and $m \in \mathbb{N}$ such that $\gamma_m(K) \le 1/2$ and $\gamma_m(L) \le 1/2$,

$$f_m(K) \le d_{BM}(K,L) f_m(L), \qquad f_m(K) \ge \frac{d_{BM}(K,L)}{2d_{BM}(K,L)-1} f_m(L). \tag{2}$$

Relations (2) don't work without restricting the homothety ratios.

Relationship with other quantities

Proposition 1

For any o-symmetric d-dimensional convex body K,

$$\mathrm{vein}(K) \leq \mathrm{ill}(K) \leq 2C(K) \leq 2\,\mathrm{coin}(K),$$

and in general

$$I(K) \leq C(K) \leq \mathrm{coin}(K).$$

Here $\mathrm{vein}(K) = \inf\left\{\sum_{i\in I} \|p_i\|_K : K \subseteq \mathrm{conv}\left\{p_i \in \mathbb{E}^d : i \in I\right\}\right\}$ denotes the vertex index of a symmetric convex body K defined by Bezdek and Litvak (2007).

Upper bounds

Proposition 2 (Rogers-type bounds)

Given $K \in \mathcal{K}^d$, $d \geq 2$, we have

$$\mathrm{coin}(K) < \begin{cases} 2^{2d+1} d(\ln d + \ln\ln d + 5) = O(4^d d \ln d), & K \text{ } o\text{-symmetric}, \\ 2^{d+1}\binom{2d}{d} d(\ln d + \ln\ln d + 5) = O(8^d d \ln d), & \text{otherwise.} \end{cases}$$

The proof uses Rogers' estimate of the translative covering density of K, Rogers-Shephard inequality on the volume of the difference body $K - K = K + (-K)$, and the Rogers-Zong inequality on the minimum number of translates of a convex body L that cover K.

Monotonicity

Lemma 1

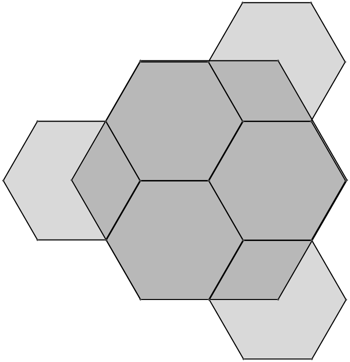
Let $j < m$ be positive integers. Then for any d-dimensional convex body K the inequality $f_j(K) > f_m(K)$ implies $m < f_j(K)$.

This shows that the covering index of any convex body can be obtained by calculating a finite minimum, rather than the infimum of an infinite set. In particular, if $f_j(K) < \infty$ for some j, then

$$\mathrm{coin}(K) = \min \{ f_m(K) : m < f_j(K) \} .$$

A simple example

The figure shows that an affine regular convex hexagon H can be covered by 6 half-sized homothets. Thus $\text{coin}(H) \leq f_6(H) \leq 12$.



A simple example

Any half-sized homothet of H can cover at the most one-sixth of the boundary of H. Therefore, $\gamma_m(H) > 1/2$, for $m = 1, \ldots, 5$ and $f_6(H) = 12$. Thus,

$$\operatorname{coin}(H) = \inf\{f_m(H) : 6 \leq m < 12\} \leq 12.$$

If $f_m(H) < 12$, for some $7 \leq m \leq 11$, then $\gamma_m(H) < \frac{12-m}{12}$, and by the definition of covering, $m \operatorname{vol}(\gamma_m(H)H) = m\gamma_m(H)^2 \operatorname{vol}(H) \geq \operatorname{vol}(H)$. Therefore,

$$m\left(\frac{12-m}{12}\right)^2 > 1,$$

which is impossible for $8 \leq m \leq 11$.

This only leaves the case $m = 7$. But Lassak (1987) showed that $\gamma_7(H) = 1/2$ and as a result, $f_m(H) = 14$. We conclude that $\operatorname{coin}(H) = 12$.

Direct vector sums

Let $\mathbb{E}^d = \mathbb{L}_1 \oplus \cdots \oplus \mathbb{L}_n$ be a decomposition of $\mathbb{E}^d$ into the direct vector sum of its linear subspaces $\mathbb{L}_i$ and let $K_i \subseteq \mathbb{L}_i$ be convex bodies. Denote the direct sum of $K_1, \ldots, K_n$ by $K_1 \oplus \cdots \oplus K_n$.

Boltyanski and Martini (2007) showed that $I(K_1 \oplus \ldots \oplus K_n) \leq \prod_{j=1}^{n} I(K_j)$, but that the equality does not hold in general. No general formula exists for $I(K_1 \oplus \ldots \oplus K_n)$ in terms of illumination of the $K_j's$.

Let us denote by $N_\lambda(K)$, the minimum number of homothets of K of homothety ratio $0 < \lambda \leq 1$ needed to cover K. Then

$$\operatorname{coin}(K) = \inf\left\{\frac{m}{1-\gamma_m(K)} : \gamma_m(K) \leq \frac{1}{2}, m \in \mathbb{N}\right\} = \inf_{\lambda \leq \frac{1}{2}} \frac{N_\lambda(K)}{1-\lambda}.$$

Direct vector sums

Theorem 1

Let $\mathbb{E}^d = \mathbb{L}_1 \oplus \cdots \oplus \mathbb{L}_n$ *be a decomposition of* $\mathbb{E}^d$ *into the direct vector sum of its linear subspaces* $\mathbb{L}_i$ *and let* $K_i \subseteq \mathbb{L}_i$ *be convex bodies such that* $\mathrm{coin}(K_i) = f_{m_i}(K_i)$, $i = 1, \ldots, n$, *and* $\Gamma = \max\{\gamma_{m_i}(K_i) : 1 \leq i \leq n\}$. *Then*

$$\max\{\mathrm{coin}(K_i) : 1 \leq i \leq n\} \leq$$

$$\mathrm{coin}(K_1 \oplus \cdots \oplus K_n) = \inf_{\lambda \leq \frac{1}{2}} \frac{\prod_{i=1}^n N_\lambda(K_i)}{1-\lambda}$$

$$\leq \frac{\prod_{i=1}^n N_\Gamma(K_i)}{1-\Gamma} \leq \frac{\prod_{i=1}^n m_i}{1-\Gamma} < \prod_{i=1}^n \mathrm{coin}(K_i). \quad (3)$$

Moreover, the first two upper bounds in (3) are tight.

1-codimensional cylinders

Let $K \subseteq \mathbb{E}^d \subseteq \mathbb{E}^d \oplus \mathbb{E}^1 = \mathbb{E}^{d+1}$ be a d-dimensional convex body and $\ell \subseteq \mathbb{E}^1 \subseteq \mathbb{E}^d \oplus \mathbb{E}^1 = \mathbb{E}^{d+1}$ denote a line segment that can be optimally covered (in the sense of coin) by two homothets of homothety ratio $1/2$. We say that the $d+1$-dimensional convex body $K \oplus \ell \subseteq \mathbb{E}^{d+1}$ is a *1-codimensional cylinder*.

Theorem 2

For any 1-codimensional cylinder $K \oplus \ell$, the first two upper bounds in (3) become equalities and

$$\operatorname{coin}(K \oplus \ell) = 4N_{1/2}(K).$$

Minkowski sums

Let Δ^d denote the d-simplex.

Theorem 3

Let the convex body K be the vector sum of the convex bodies $K_1, \ldots, K_n$ in $\mathbb{E}^d$, i.e., let $K = K_1 + \cdots + K_n$, such that $\operatorname{coin}(K_i) = f_{m_i}(K_i)$, $i = 1, \ldots, n$, and $\Gamma = \max\{\gamma_{m_i}(K_i) : 1 \leq i \leq n\}$. Then

$$\operatorname{coin}(K) = \operatorname{coin}(K_1 + \cdots + K_n) \leq \frac{\prod_{i=1}^{n} N_\Gamma(K_i)}{1-\Gamma} \leq \frac{\prod_{i=1}^{n} m_i}{1-\Gamma} < \prod_{i=1}^{n} \operatorname{coin}(K_i). \tag{4}$$

Moreover, equality in (4) does not hold in general.
(E.g., $H = \Delta^2 + (-\Delta^2)$, but $\operatorname{coin}(\Delta^2) = \operatorname{coin}(H) = 12$.)

Continuity

Theorem 4

Let $\mathcal{K}_m^d := \{K \in \mathcal{K}^d : \gamma_m(K) \le 1/2\}$.
(i) For any $K, L \in \mathcal{K}_m^d$,

$$f_m(K) \le d_{BM}(K,L) f_m(L), \qquad f_m(K) \ge \frac{d_{BM}(K,L)}{2d_{BM}(K,L) - 1} f_m(L).$$

(ii) The functional $f_m : \mathcal{K}_m^d \longrightarrow \mathbb{R}$ *is* **uniformly continuous** *for all* d *and* m.
(iii) Define $I_K = \{i : \gamma_i(K) \le 1/2\} = \{i : K \in \mathcal{K}_i^d\}$, *for any* d*-dimensional convex body* K. *If* $I_L \subseteq I_K$, *for some* $K, L \in \mathcal{K}^d$, *then (but not in general)*

$$\operatorname{coin}(K) \le d_{BM}(K,L) \operatorname{coin}(L), \qquad \operatorname{coin}(K) \le \frac{2d_{BM}(K,L) - 1}{d_{BM}(K,L)} \operatorname{coin}(L).$$

(iv) The functional $\operatorname{coin} : \mathcal{K}^d \longrightarrow \mathbb{R}$ *is* **lower semicontinuous** *for all* d.

Optimizers

Theorem 5

Let B^d *and* C^d *denote the d-dimensional ball and d-dimensional cube, respectively.*

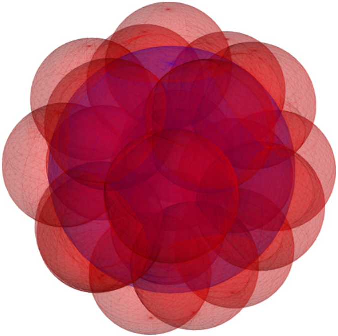
(i) For any $K \in \mathcal{K}^d$, $\mathrm{coin}(C^d) = 2^{d+1} \leq \mathrm{coin}(K)$ *and so affine d-cubes minimize the covering index in all dimensions.*

(ii) If K *is a planar convex body then* $\mathrm{coin}(K) \leq \mathrm{coin}(B^2) = 14$.

(iii) If $K \oplus \ell$ *is a 1-codimensional cylinder in* $\mathcal{K}^3$, *then* $\mathrm{coin}(K \oplus \ell) \leq 28$, *that is* $B^2 \oplus \ell$ *maximizes* coin *among 1-codimensional 3-dimensional cylinders.*

Another example: $\mathrm{coin}(B^d), d \geq 3$

Recently, O'Rourke (2012) raised the question as to what is the minimum number of homothets of homothety ratio $1/2$ needed to cover B^3.



Using spherical cap coverings, Wynn (2012) showed this number to be 21. In fact, if the homothety ratio is decreased to 0.49439, we can still cover B^3 by 21 homothets. Therefore, $\mathrm{coin}(B^3) \leq 41.53398\ldots$.

Verger-Gaugry (2005) showed that in any dimension $d \geq 2$ one can cover a ball of radius $1/2 < r \leq 1$ with $O((2r)^{d-1} d^{3/2} \ln d)$ balls of radius $1/2$. This shows that $\mathrm{coin}(B^d) = O(2^d d^{3/2} \ln d)$.

Known values/estimates

K	m	$\gamma_m(K)$	$\mathrm{coin}(K)$
ℓ	2	$1/2$	4
H	6	$1/2$	12
Δ^2	6	$1/2$	12
B^2	7	$1/2$	14
B^3	21	≤ 0.49439	$\leq 41.53398\ldots$
B^d	$O(2^d d^{3/2} \ln d)$	$\leq 1/2$	$O(2^d d^{3/2} \ln d)$
C^d	2^d	$1/2$	2^{d+1}
$H \oplus \ell$	12	$1/2$	24
$\Delta^2 \oplus \ell$	12	$1/2$	24
$B^2 \oplus \ell$	14	$1/2$	28

Open questions

Problem 1 (Upper semicontinuity)

Either prove that coin *is upper semicontinuous or construct a counterexample.*

Since circles maximize coin in the plane, it is reasonable to propose the following.

Problem 2 (Maximizers)

For any d-dimensional convex body K, prove or disprove that $\mathrm{coin}(K) \leq \mathrm{coin}(B^d)$ *holds.*

An affirmative answer would considerably improve the known general (Rogers-type) upper bound on the illumination number from $O(4^d d \ln d)$ to $O(2^d d^{3/2} \log d)$.

References

K. Bezdek and M. A. Khan, The covering index of convex bodies, *preprint*.

Centre for Computational and Discrete Geometry `http://math.ucalgary.ca/ccdg`.