Discrete Geometry and Symmetry

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Symmetry is among the most frequently recurring themes in the natural sciences and has inspired much of modern geometry.

Highly symmetric figures that are discrete naturally fall into the realm of discrete geometry. In a broad sense, discrete geometry investigates discrete structures in geometry and combinatorics such as polytopes, polyhedra and maps, tessellations (tilings), complexes and graphs, efficient sphere arrangements, packing and covering arrangements, and lattices. The Workshop focused on aspects of symmetry in the analysis and classification of such structures, and exploited symmetry as a unifying theme. The Workshop brought together established experts and emerging researchers in discrete geometry and related areas, to share recent developments in the study of highly-symmetric discrete geometric structures, discuss emerging directions, encourage new collaborative ventures, and achieve further progress on fundamental questions in the field.

The last few decades have seen a revival of interest in discrete geometry and symmetry, and have produced groundbreaking new discoveries. The Workshop presented new directions of research, explored new approaches to old problems, and helped formulate conjectures pointing to a fascinating world of highly-symmetric structures still to be discovered. The themes of the Workshop reflected the rich mathematical traditions of the groundbreaking works of H.S.M. Coxeter, László Fejes Toth, Branko Grünbaum and Peter McMullen to whom we owe a great deal of our present understanding of discrete geometry, and the more recent progress in the field that they inspired. The 50th anniversary in 2014 of the publication of Fejes Tóth's classic on "Regular Figures" by Pergamon Press in 1964, as well as the upcoming Fejes Toth Centennial celebrations in Budapest in 2015, made the Workshop a particularly timely event.

In the area of polytope-like structures and symmetry, much of the recent progress has centered around the modern theory of abstract polytopes and combinatorial symmetry [12]. Abstract polytopes are combinatorial structures with distinctive geometric, algebraic, or topological properties, in many ways more fascinating than traditional polyhedra, polytopes and tessellations. While much of the fundamental work of Coxeter and Grünbaum in this area focused on highly regular structures, recent research also dealt with somewhat less restricted aspects of symmetry, thus broadening the classical approach while leading to many new and unexplored problems. There has been much recent research on the chirality in polyhedra, maps or polytopes. Chirality is a fascinating phenomenon which does not occur in the traditional theory of polyhedra and polytopes.

The rapid development of abstract polytope theory has resulted in a rich theory featuring an attractive interplay of methods and tools from discrete geometry (classical polytope theory), group theory and geometry (Coxeter groups and their quotients, as well as reflection and crystallographic groups), combinatorial group theory (generators and relations), and hyperbolic geometry and topology (tessellations and their groups). Still, even after an active period of research, many deep problems have remained open and await solution.

Efficient packings of solids have been investigated since the times of Kepler. He was the first to formulate the discrete geometric problem of finding the densest packings of spheres (balls) in ordinary space, and his conjecture about the most efficient arrangement became known as the Kepler Conjecture (finally confirmed by Hales). The systematic research on general packing and covering problems in space began in the late 1940's with the pioneering work of Fejes Tóth. The Hungarian geometry school of Fejes Toth greatly contributed to the growing field of discrete geometry and has attracted the interest of numerous other mathematicians, including prominent researchers such as Coxeter, Rogers, Penrose, and Conway.

Two particularly active subject areas, already highlighted in Hilbert's 18th Problem, stand out since the early days of discrete geometry and are naturally intertwined, namely dense sphere packings, and tiling theory. The interest in sphere packings generated a great deal of research on the geometry of Voronoi tilings with a staggering number of real world applications. Over the past few years, it has become increasingly evident that further progress on most "hard problems" about efficient arrangements of solids would also require new optimization techniques [1, 8]. Many central and by now classical problems in discrete geometry have an established record of strong connections with geometric analysis, coding theory, group theory (symmetry groups), number theory, and differential and integral geometry. The connections with combinatorics and optimization are of particular importance and have not yet been fully exploited. Advanced optimization techniques have significantly helped achieve recent breakthrough results on kissing numbers and densest lattice sphere packings [4, 13].

1 Discrete Structures and Symmetry

1.1 Polyhedra and Polytopes

Nikolay Abrosimov, Sobolev Institute of Mathematics, Novosibirsk, presented joint work with Alexander Mednykh and Ekaterina Kudina about the volume of hyperbolic octahedra with $\overline{3}$-symmetry, to appear in Proceedings of Steklov Institute of Mathematics. An octahedron is said to have $\overline{3}$-symmetry if its symmetries include a rotation of order 3 as well as he antipodal involution.

Abrosimov first discussed the Euclidean case exploiting intuition, and in particular derived both an existence criterion for an octahedron with $\overline{3}$-symmetry as well as a volume formula. Embedding the Euclidean octahedron in the projective Cayley-Klein model of hyperbolic 3-space then permitted to compute the hyperbolic edge lengths and dihedral angles in terms of the coordinates of the vertices; these parameters are non-invariant and depend on the choice of coordinate chart. Then the coordinates were eliminated and relationships between the edge lengths and the dihedral angles were obtained. Using this relationships, a Schläfli equation was solved and an explicit volume formula for a hyperbolic octahedron with $\overline{3}$-symmetry was presented. In addition, an existence criterion for a hyperbolic octahedron with $\overline{3}$-symmetry was described.

Javier Bracho, UNAM at Mexico City, discussed highly symmetric realizations of abstract polytopes as geometric polytopes in Euclidean spaces. A geometric polytope is geometrically regular if it has a flag-transitive geometric symmetry group; the polytope then has maximum possible symmetry by reflection. A geometric polytope is geometrically chiral if it has a geometric symmetry group with two orbit on the flags such that adjacent flags are in distinct orbits; the polytope then has maximum possible symmetry by rotation.

Bracho presented an unexpected example of a finite regular abstract polytope of rank 4 with a chiral embedding in $\mathbb{E}^4$, that is, a geometrically chiral 4-polytopes in $\mathbb{E}^4$. This disproved an earlier claim in the literature that no such geometric polytopes exist.

Frieder Ladisch, University of Rostock, gave a talk about affine symmetries of orbit polytopes. An orbit polytope is the convex hull of a point orbit under a finite subgroup G of $\mathrm{GL}(\mathrm{d}, \mathbb{R})$. Ladisch studied the possible affine symmetry groups of orbit polytopes. For every group, there is an open and dense set of "generic points" such that the orbit polytopes of generic points have conjugate affine symmetry groups and are minimal in a certain sense. The symmetry group of a generic orbit polytope coincides with G if G is itself the affine symmetry group of some orbit polytope, or if G is absolutely irreducible. On the other hand, there are some general cases where the affine symmetry group grows, for example representation polytopes (the convex hull of a finite matrix group). Their affine symmetries can be computed effectively from a certain character. The results presented were joint work with Erik Friese.

Nicholas Matteo, Northeastern University, discussed the classification of convex polytopes with few flag orbits under the geometric symmetry group action. The convex polytopes with a single flag orbit are precisely the regular convex polytopes. In earlier work, Matteo had classified the two-orbit convex polytopes (as well as the convex polytopes which have two flag orbits under the combinatorial automorphism group). In his talk, Matteo described a full classification of the convex polytopes with three flag orbits under the symmetry group. These polytopes exist only in eight dimensions or fewer. Tilings of Euclidean spaces with few flag orbits under the geometric symmetry group were also described. The results presented have appeared in Matteo's 2015 PhD thesis on "Convex Polytopes and Tilings with Few Flag Orbits" at Northeastern University.

Abigail Williams, Northeastern University, gave a lecture about uniform skeletal polyhedra in ordinary 3-space. In skeletal polyhedra, each face is considered to be a set of edges which is not spanned by a membrane as in traditional convex polyhedra. The faces, and indeed the polyhedra themselves, are hollow. The uniformity condition signifies that the polyhedra have regular faces and are vertex transitive under the geometric symmetry group. Williams described a construction which can be used to generate uniform skeletal polyhedra from the symmetry groups of the regular skeletal polyhedra. Also discussed was an extension of this construction which can be used to generate more uniform skeletal polyhedra. The results presented have appeared in Williams' 2015 PhD thesis on "Wythoffian Skeletal Polyhedra" at Northeastern University.

1.2 Surfaces, maps, and graphs

Marston Conder, University of Auckland, kicked off the workshop, with a talk on arc-types of vertex-transitive graphs. Let X be vertex-transitive graph of valency d, and let A be its full automorphism group. Then the *arc-type* of X is defined in terms of the lengths of the orbits of the action of the stabiliser A_v of a given vertex v on the set of arcs incident with v. Specifically, the arc-type is the partition of d as the sum $n_1 + n_2 + \cdots + n_t + (m_1 + m_1) + (m_2 + m_2) + \cdots + (m_s + m_s)$, where $n_1, n_2, \ldots, n_t$ are the lengths of the self-paired orbits, and $m_1, m_1, m_2, m_2, \ldots, m_s, m_s$ are the lengths of the non-self-paired orbits, in ascending order. For example, if X is arc-transitive then its arc-type is d, while if X is half-arc-transitive then its arc-type is $d/2+d/2$. In his talk Conder explained how it can be shown that there are vertex-transitive graphs with every possible arc-type, except $1 + 1$ and $(1 + 1)$.

Undine Leopold, Technical University of Chemnitz, presented Part I of a joint talk with Tom Tucker on euclidean symmetry of closed surfaces immersed in 3-space. Given a finite group G of orientation-preserving euclidean isometries and a closed surface S, an immersion $f : S \to E^3$ is in G-general position if $f(S)$ is invariant under G, points of S have disk neighborhoods whose images are in general position, and no singular points of $f(S)$ lie on an axis of rotation of G. For such an immersion, there is an induced action of G on S whose Riemann-Hurwitz equation satisfies certain natural restrictions.

In the first part of this talk, Leopold introduced these restrictions and presented how models arise from the quotient surface S/G in the orbifold E^3/G. It may be particularly surprising that an orientable symmetric surface can lead to a nonorientable quotient. Leopold also pointed out that the problem of classifying which of the restricted Riemann-Hurwitz equations are realizable becomes intractable outside of G-general immersions.

Thomas W. Tucker, Colgate University, gave Part II of a joint talk with Undine Leopold on euclidean symmetry of closed surfaces immersed in 3-space. In the second part of this talk, Tucker focussed on additional group theoretic conditions that must be satisfied by G and the fundamental groups of the surface S and its quotient surface, before completing the classification of which restricted Riemann-Hurwitz equations are realized by a G-general position immersion of S. Exceptions arise, in particular, for low genus and little branching. One is then able to decide which genera of a surface allow a G-general immersion in 3-space.

1.3 Abstract Polytopes and Groups

Eric Ens, York University, discussed consistent colourings of polytopes. A colouring of the facets of a polytope is called consistent if the colouring is respected (though not necessarily preserved) by the automorphism group. Any polytope can be coloured trivially be assigned a different colour to each facet or by assigning the same colour to each facet. Interesting examples of consistent colourings were discussed, and then colourings of the regular and chiral toroidal polytopes of type $\{4, 4\}$ were examined in more depth.

Isabel Hubard, UNAM at Mexico City, lectured about products of abstract polytopes. Given two convex polytopes the operations of taking their join, their cartesian product, and their direct sum are well understood. In her talk, Hubard described how these three kinds of products can be extended to abstract polytopes. She also introduced a new product, called the topological product, which also arises in a natural way from geometry.

One is particularly interested in understanding the automorphism group of a product of $\mathcal{P}$ and $\mathcal{Q}$ in terms of the automorphism groups of $\mathcal{P}$ and $\mathcal{Q}$. To this end, Hubard introduced the concept of a *prime* polytope, for a given product. We shall see that highly symmetric non-prime polytopes are sparse; in fact, for the join product the only regular non-prime polytopes are the simplices, for the cartesian product the only regular non-prime polytopes are the hypercubes, for the direct sum the only regular non-prime polytopes are the cross polytopes and for the topological product the only regular non-prime polytopes are toroidal polytopes.

Kyle Meyer, Northeastern University, discussed face enumeration for the colorful associahedra and related structures. The classical associahedra can be formulated in terms of flipping the diagonals of triangulations of convex polygons. Similarly the colorful associahedra, an abstract polytope, introduced by Araujo-Pardo, Hubard, Oliveros, and Schulte, is formulated in terms of flipping diagonals of triangulations whose diagonals are colored (colored triangulations). In this talk, Meyer gave a modified formulation of the colorful associahedra in terms of partial colored triangulations, and using this formulation counted the number of faces of the colorful associahedra by dimension.

Egon Schulte, Northeastern University, gave a survey talk about colorful polytopes, associahedra and cyclohedra. Every n-edge colored n-regular graph G naturally gives rise to a simple abstract n-polytope $P(G)$, called the colorful polytope of G, whose 1-skeleton is isomorphic to G. Schulte described colorful polytope versions of the associahedron and cyclohedron. Like their classical counterparts, the colorful associahedron and cyclohedron encode triangulations and flips, but now with the added feature that the diagonals of the triangulations are colored and adjacency of triangulations requires color preserving flips. The colorful associahedron and cyclohedron are derived as colorful polytopes from the edge colored graph whose vertices represent these triangulations and whose colors on edges represent the colors of flipped diagonals. This was joint work with G.Araujo-Pardo, I.Hubard and D.Oliveros.

Micael Toledo, UNAM at Mexico City, spoke about the automorphism groups and the symmetry type graphs of maniplexes. A manicomplex is a generalization of an abstract polytope, in much the same way in which a map on a surface is a generalization of an abstract polyhedron. For a given maniplex M let O denote its set of flag orbits under the action of the automorphism group. Then an edge-coloured graph with vertex set O can be constructed by joining two vertices o_1 and o_2 by an i-coloured edge whenever there are flags f in o_1 and g in o_2 which are i-adjacent. This graph is called the symmetry type graph of M. In this talk, Toledo discussed symmetry type graphs of maniplexes. In particular, given a symmetry type graph, generators for the automorphism group of a maniplex with this symmetry type graph were presented

1.4 Polytopes and Incidence Geometries

Maria Elisa Carrancho Fernandes, University of Aveiro, talked about regular and chiral hypertopes. In 1983, Aschbacher proved that string C-groups are thin, residually connected, regular geometries. The talk concerned C-groups with nonlinear Coxeter diagrams. It was shown that thin, residually connected regular geometries are C-groups, but that the converse is not true. Nevertheless flag-transitivity is a sufficient condition to establish the converse: flag-transitive C-groups are thin, residually connected regular geometries (the Tits algorithm is used to get an incidence geometry from a C-group).

Abstract regular polytopes are string C-groups, as described by McMullen and Schulte in their book (2002). For this reason, the term (regular) hypertope is used to designate a thin, residually connected (regular) geometry. Abstract regular polytopes are regular hypertopes with linear Coxeter diagram. Guided by the ideas of chirality in the abstract polytope theory, we extend the concept to a more general setting of incidence geometries. Indeed, when the geometry is thin, it is possible to define chirality, as in the case of polytopes. We give characterisations of automorphism groups of thin residually connected chiral geometries and we show how to construct such chiral objects group-theoretically. One of our focus is the classification of hypertopes of a certain type. Here we consider spherical, locally spherical and locally toroidal hypertopes (hypertopes having all parabolic subgroups either spherical or toroidal).

Dimitri Leemans, University of Auckland, discussed the classification of abstract polytopes whose automorphism group is an almost simple group of PSL(2,q) type. The talk explained the classification of the regular polytopes for the groups PSL(2,q) and PGL(2,q) obtained in joint work with Egon Schulte, and then elaborated on more general results established jointly with Thomas Connor and Julie De Saedeleer. Leemans also described the current state of the classification of the chiral polytopes related to these groups; this is ongoing work with Eugenia OReilly-Regueiro and Jeremie Moerenhout.

Eugenia O'Reilly-Regueiro, UNAM at Mexico City, continued the theme of Leemans' talk and spoke about "Abstract polytopes and projective lines, the chiral case". The classification of abstract polytopes with almost simple automorphism group of PSL(2,q) type has been addressed separately for the regular and the chiral cases. The regular case was presented by Dimitri Leemans; it was completed jointly with Thomas Connor and Julie De Saedeleer following previous work with Egon Schulte. This talk presented some results on the chiral case, from ongoing joint work with Dimitri Leemans and Jeremie Moerenhout.

2 Discrete Convex Geometry

2.1 Convex Geometry

Ryan Trelford, University of Calgary and York University, spoke about X-raying of 3-dimensional convex bodies with mirror symmetry. Let K be a d-dimensional convex body. A point p on the boundary of K is said to be X-rayed along a line with direction vector $\mathbf{v}$ if the line through p with direction $\mathbf{v}$ intersects the interior of K. A collection of lines is said to X-ray K if every boundary point of K is X-rayed along one of the lines. The minimum number of lines required to X-ray K is called the X-ray number of K, and is denoted by $X(K)$. In 1994, K.Bezdek and T.Zamfirescu conjectured that $X(K) \leq 3 \cdot 2^{d-2}$ for any d-dimensional convex body K.

The talk explained how the X-ray Conjecture is related to the famous Gohberg-Markus-Hadwiger Covering Conjecture. Trelford briefly verified the X-ray conjecture for planar convex bodies, showing that three lines are needed if, and only if, the convex body is a triangle. Then it was proved that any 3-dimensional convex body exhibiting mirror symmetry also satisfies the X-ray Conjecture.

Vlad Yaskin, University of Alberta, discussed stability results for sections of convex bodies. Let K be a convex body in $\mathbb{R}^n$. The *parallel section function* of K in the direction $\xi \in S^{n-1}$ is defined by

$$A_{K,\xi}(t) = \mathrm{vol}_{n-1}(K \cap \{\xi^{\perp} + t\xi\}), \quad t \in \mathbb{R}.$$

If K is origin-symmetric (i.e. $K = -K$), then Brunn's theorem implies

$$A_{K,\xi}(0) = \max_{t\in\mathbb{R}} A_{K,\xi}(t)$$

for all $\xi \in S^{n-1}$.

The converse statement was proved by Makai, Martini and Ódor. Namely, if $A_{K,\xi}(0) = \max_{t\in\mathbb{R}} A_{K,\xi}(t)$ for all $\xi \in S^{n-1}$, then K is origin-symmetric.

Yaskin, in joint work with Matthew Stephen, provided a stability version of this result. If $A_{K,\xi}(0)$ is close to $\max_{t\in\mathbb{R}} A_{K,\xi}(t)$ for all $\xi \in S^{n-1}$, then K is close to $-K$.

2.2 Packing and Covering

Karoly Bezdek, University of Calgary, gave a survey about contact numbers, summarizing old and new results. Contact numbers are natural extensions of kissing numbers. The talk focussed on estimating the contact numbers in a packing of n unit balls in Euclidean d-space.

Muhammad Khan, University of Calgary, spoke about joint work with Karoly Bezdek on the covering index of convex bodies. Covering a convex body by its homothets is a classical notion in discrete geometry that has resulted in a number of interesting and long standing problems. Swanepoel introduced the covering parameter of a convex body as a means of quantifying its covering properties. Khan introduced a relative of the covering parameter called covering index, which turns out to have a number of nice properties. Intuitively, the covering index measures how well a convex body can be covered by a relatively small number of homothets having a relatively small homothety ratio. It was shown that the covering index provides a useful upper bound for well-studied quantities like the illumination number, the illumination parameter, the vertex index and the covering parameter of a convex body. Khan obtained upper bounds on the covering index and investigated its optimizers. Furthermore, it was shown that the covering index satisfies a nice compatibility with the operations of direct vector sum and vector sum that helps in determining the covering index of several convex bodies.

Marton Naszodi, Ecole Polytechnique Federale de Lausanne, and Eötvös Loránd University, gave a lecture about coverings in Euclidean space and on the sphere. The talk presented a method to obtain upper bounds on covering numbers. As applications of this method, Naszodi reproved and generalized results of Rogers on economically covering Euclidean n-space (resp. the sphere) with translates resp. rotated copies of a (spherically) convex body, or more generally, any measurable set. Using the same method, Naszodi sharpened an estimate by Artstein–Avidan and Slomka on covering a bounded set by translates of another.

The main novelty of the method described is that it was not probabilistic. The key idea, which made the proofs rather simple, is an algorithmic result of Lovász.

2.3 Convex Polytopes

Wendy Finbow-Singh, St. Mary's University, presented a talk on low dimensional neighbourly simplicial polytopes. Amongst the d-polytopes with v vertices, the neighbourly polytopes have the greatest number of facets. This maximum property has prompted researchers to compose lists of them. Finbow-Singh discussed an algorithm for generating the list of simplicial neighbourly d-polytopes with v vertices, for a given dimension d and number of vertices, v.

Alexander Litvak, University of Alberta, lectured about joint work with D. Alonso-Gutierrez and Nicole Tomczak-Jaegermann on the isotropic constant of random polytopes. Let $X_1, \ldots, X_N$ be independent random vectors uniformly distributed on an isotropic convex body $K \subset \mathbb{R}^n$, and let K_N be the symmetric convex hull of X_i's. Litvak showed that with high probability $L_{K_N} \leq C\sqrt{\log(2N/n)}$, where C is an absolute constant. This result closed the gap in known estimates in the range $Cn \leq N \leq n^{1+\delta}$. Furthermore, Litvak extended the estimates to the symmetric convex hulls of vectors $y_1X_1, \ldots, y_NX_N$, where $y = (y_1, \ldots, y_N)$ is a vector in $\mathbb{R}^N$. Also discussed was the case of a random vector y.

2.4 Graph Drawings

Janos Pach, Ecole Polytechnique Federale de Lausanne, and Renyi Institute, gave a lecture on the number of crossings between curves.

David Richter, Western Michigan University, talked about algebraic universality of parallel drawings. Let Σ be a set of d fixed-point-free involutions on a given set $S = \{1, 2, 3, ..., 2n\}$. Graph-theoretically, this is the same as specifying a d-regular multigraph with vertex set S and an edge coloring by d colors. A *parallel drawing* of Σ is a drawing of the underlying graph in which every edge is represented by a segment and the segments sharing a common color are mutually parallel. The purpose of this talk was to explain "algebraic universality" for parallel drawings in the plane in the case when $|\Sigma| = 4$.

2.5 Helly's Theorem and Relatives

Deborah Oliveros, UNAM at Queretaro, spoke about joint work with J.A. De Loera, R.N. La Haye and E. Roldán-Pensado about Helly's Theorem over subgroups and other additive subsets of $\mathbb{R}^d$. In the usual Helly-type theorems, the convex sets are required to intersect in a proper subset S of $\mathbb{R}^d$. For instance, in the classical Helly's theorem this subset is $S = \mathbb{R}^d$ and the Helly number is $d + 1$; and for Doignon's theorem, S is the set of integer points $\mathbb{Z}^d$ and the Helly number is 2^d. Oliveros presented some extensions of these results to the case when S is an arbitrary additive subgroup of $\mathbb{R}^d$, as well as some other interesting related results in dimension 2.

3 Convex and Combinatorial Geometry Fest

The 5-day Discrete Geometry and Symmetry Workshop was directly followed by the 2-day Convex and Combinatorial Geometry Fest at BIRS (15w2177), February 13-15, 2015. Organizer of this event were Abhinav Kumar (MIT), Daniel Pellicer (Universidad Nacional Autonoma de Mexico), Konrad Swanepoel (London School of Economics and Political Science), and Asia Ivić Weiss (York University). The programs of the two workshops were coordinated.

The 2-day Workshop explored the ways in which the areas of abstract polytopes and discrete convex geometry has been influenced by the work of Karoly Bezdek and Egon Schulte. The last three decades have witnessed the revival of interest in these subjects and great progress has been made on many fundamental problems. The Workshop was held to honor the occasion of Bezdek's and Schulte's 60-th birthday.

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