Local maps and representation theory

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DEFINITION If $\mathcal{X}$ is Banach space and $\mathcal{S} \subseteq B(\mathcal{X})$, then $\mathcal{S}$ is said to be reflexive iff the following condition is satisfied

$$T \in B(\mathcal{X}),\ Tx \in [\mathcal{S}x], \forall x \in \mathcal{X} \implies T \in \mathcal{S}$$

For $\mathcal{S} \subseteq X$ unital algebra this is equivalent to the familiar

$$T(M) \subseteq M, \forall M \in \mathrm{Lat}\, \mathcal{S} \implies T \in \mathcal{S}$$

THEOREM 1. (Katsoulis, 2014) Let E a Hilbert C*-module over a C*-algebra $\mathcal{A}$ and let $L(E)$ be the adjointable operators on E. Then

$$\operatorname{Lat} L(E) = \{E\mathcal{J} \mid \mathcal{J} \subseteq \overline{\langle E, E\rangle} \text{ closed left ideal } \}$$

where

$$E\mathcal{J} = \{\xi j \mid \xi \in E, j \in \mathcal{J}\}.$$

and the association $\mathcal{J} \mapsto E\mathcal{J}$ establishes a complete lattice isomorphism between the closed left ideals of $\overline{\langle E, E\rangle}$ and $\operatorname{Lat} L(E)$.

Note that if $\operatorname{End}_{\mathcal{A}}(E)$ denotes the bounded $\mathcal{A}$-module opertors on E, then the above implies

$$\operatorname{Lat} L(E) = \operatorname{Lat} \operatorname{End}_{\mathcal{A}}(E)$$

THEOREM (Katsoulis 2014). Let E be a Hilbert module over a C*-algebra $\mathcal{A}$. Then

$$\mathsf{Alg}\,\mathsf{Lat}\,\mathcal{L}(E) = \mathsf{End}_{\mathcal{A}}(E).$$

In particular, $\mathsf{End}_{\mathcal{A}}(E)$ is a reflexive algebra of operators acting on E.

The proof follows from the following

THEOREM. (Johnson 1968) Let $\mathcal{A}$ be a semisimple Banach algebra and let Φ be a linear operator acting on $\mathcal{A}$ that leaves invariant all closed left ideals of $\mathcal{A}$. Then

$$\Phi(ba) = \Phi(b)a, \forall\, a, b \in \mathcal{A},$$

i.e, Φ is a left multiplier.

In particular, if $1 \in \mathcal{A}$ is a unit then Φ is the left multiplication operator by $\Phi(1)$.

DEFINITION. A map $S : \mathcal{A} \to \mathcal{X}$ into a Banach $\mathcal{A}$-module is said to be a local left multiplier iff for every $A \in \mathcal{A}$, there exists left multiplier $\Phi_A \in LM(\mathcal{A}, \mathcal{X})$ so that $S(A) = \Phi_A(A)$.

Approximate local multipliers are defined to satisfy an approximate version of the above definition.

PROPOSITION. Let $\mathcal{A}$ be a Banach algebra with approximate unit. If $S \in B(\mathcal{A})$, then the following are equivalent

(i) S is a closed left ideal preserver

(ii) S is an approximate local left multiplier

(iii $S \in \mathrm{Alg}\,\mathrm{Lat}\, LM(\mathcal{A})$.

THEOREM. (Johnson 1969). The space $LM(\mathcal{A})$ of left multipliers over a semisimple Banach algebra is reflexive, i.e.,$\mathrm{Alg\,Lat}\, LM(\mathcal{A}) = LM(\mathcal{A})$.

QUESTION What about Johnson's Theorem in the context of non-semisimple operator algebras?

The answer: in general not valid!

PROPOSITION. (Hadwin 90's) Let $\mathcal{A}$ be a Banach algebra generated by its idempotents and $\mathcal{X}$ be a right Banach $\mathcal{A}$-module. Then any approximately local left multiplier from $\mathcal{A}$ into $\mathcal{X}$ is a multiplier. Hence $LM(\mathcal{A}, \mathcal{X})$ is reflexive.

Proof. Let $S : \mathcal{A} \to \mathcal{X}$ be an approximate left multiplier. Note that for any $A, P \in \mathcal{A}$ with $P = P^2$, we have $S(AP) \in \overline{\mathcal{X}P}$ and $S(A(I - P)) \in \overline{\mathcal{X}(I-P)}$ Therefore

$$\begin{aligned} S(A)P &= S(AP)P + S(A(I-P))P \\ &= S(AP)P = S(AP). \end{aligned}$$

Repeated applications of the above establish the result in the case where P is a product of idempotents. Since $\mathcal{A}$ is generated by such, S is a left multiplier, as desired.

The previous result formed the basis for a variety of results by Hadwin, Li, Pan Dong and others to establish reflexivity for $LM(A)$, where $\mathcal{A}$ ranges over a variety of algebras rich in idempotents, including nest and CSL algebras.

QUESTION. What about semicrossed products? Or tensor algebras of multivariable systems? What about their spaces of derivations? Are they reflexive? Are local derivations actually derivations? (Kadison, Larson, Sorour, Johnson, Samei)

DRAWBACK. Such algebras might contain no idempotents :(

ALTERNATE APPROACH. Representation theory

THEOREM. (Katsoulis 2014) If $\mathcal{G} = (\mathcal{G}^0, \mathcal{G}^1, r, s)$ is a topological graph, then the finite dimensional nest representations of its tensor algebra $\mathcal{T}_{\mathcal{G}}^+$ separate points.

The above generalizes an earlier result of Davidson and Katsoulis (2006) regarding tensor algebras of graphs.

COROLLARY. If $\mathcal{G} = (\mathcal{G}^0, \mathcal{G}^1, r, s)$ is a topological graph, then $LM\left(\mathcal{T}_{\mathcal{G}}^{+}\right)$ is reflexive.

Proof. Let S be an approximately local multiplier on $\mathcal{T}_{\mathcal{G}}^{+}$ and let

$$\rho_i : \mathcal{T}_{\mathcal{G}}^{+} \to B(\mathcal{H}_i), \quad i \in \mathbb{I}$$

separating family of representations on finite dimensional Hilbert space so that each $\rho_i(\mathcal{T}_{\mathcal{G}}^{+})$ is a finite dimensional nest algebra. Furthermore, $\rho_i(\mathcal{T}_{\mathcal{G}}^{+})$ is a right $\mathcal{T}_{\mathcal{G}}^{+} / \ker \rho_i$-module, with the right action coming from ρ_i. Since S preserves closed left ideals we obtain

$$S_i : \mathcal{T}_{\mathcal{G}}^{+} / \ker \rho_i \longrightarrow \rho_i(\mathcal{T}_{\mathcal{G}}^{+}); A + \ker \rho_i \longmapsto \rho_i(S(A)).$$

However S_i is an an approximate left multiplier and so Hadwin'sTheorem implies that S_i is a actually a left multiplier. Hence

$$S_i(AB + \ker \rho_i) = S_i(A + \ker \rho_i)\rho_i(B)$$

and so

$$\rho_i\Big(S(AB) - S(A)B\Big) = 0, \quad \text{for all } i \in \mathbb{I}.$$

Since $\cap_i \ker \rho_i = \{0\}$, the conclusion follows.

COROLLARY. A local left multiplier on $C(X) \times_\sigma \mathbb{Z}^+$ is actually a multiplier.

QUESTION Is the same true for $\mathcal{A} \times_\sigma \mathbb{Z}^+$, $\mathcal{A}$ non-commutative C*-algebra?

REMARK. The above Corollary not valid for multipliers taking valus in a $C(X)\times_\sigma\mathbb{Z}^+$-module.

THEOREM (Katsoulis 2014). Let $\mathcal{G} = (\mathcal{G}^0, \mathcal{G}^1, r, s)$ be a topological graph and let $\{\mathcal{G}_v\}_{v \in \mathcal{G}^0}$ be the family of discrete graphs associated with $\mathcal{G}$. Assume that the set of all points $v \in \mathcal{G}^0$ for which $\mathcal{G}_v$ is either acyclic or transitive, is dense in $\mathcal{G}^0$. Then any approximately local derivation on $\mathcal{T}_{\mathcal{G}}^+$ is a derivation.

COROLLARY. Let (X, σ) be a dynamical system for which the eventualy periodic points have empty interior, e.g., σ is a homeomorphism. Then any local derivation on $C(X) \times_\sigma \mathbb{Z}^+$ is actually a derivation.

QUESTION What if σ is arbitrary selfmap?