Modulation equations in dispersive continuum mechanics : application to the `hard spheres' case

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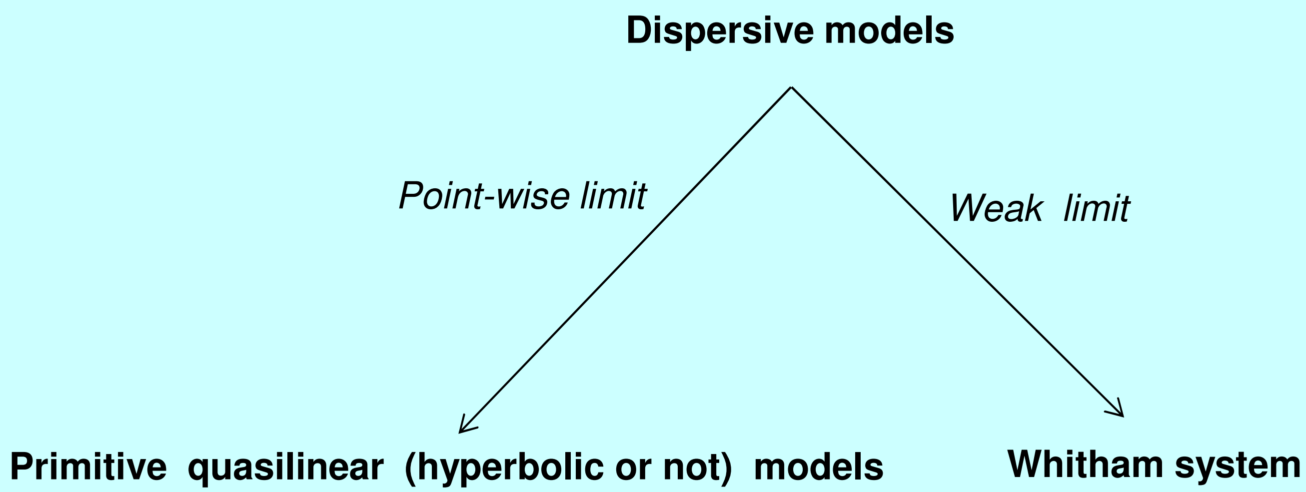
Introduction



Euler equations in Eulerian coordinates

$$\rho_t + (\rho v)_x = 0,$$
$$(\rho v)_t + \left(\rho v^2 + p(\rho)\right)_x = 0.$$

Lagrangian coordinates

$$\frac{dx}{dt} = v(t, x),$$
$$x\big|_{t=0} = X.$$

Mass Lagrangian coordinates

$$q = \int_0^X \rho_0(\xi) d\xi.$$

$$dx = x_t dt + x_X dX = v dt + \frac{\rho_0}{\rho} dX = v dt + w dq = x_t dt + x_q dq$$

p-system

$$w_t - v_q = 0,$$
$$v_t + p_q = 0, \quad p'(w) < 0, \quad w = 1/\rho.$$

σ − system

$$w_t - v_q = 0,$$
$$v_t - \sigma_q = 0, \quad \sigma'(w) > 0.$$

Additional conservation laws

$$\left(e + \frac{v^2}{2}\right)_t - (\sigma v)_q = 0, \quad \sigma = \frac{de}{dw}$$
$$(vw)_t - \left(\frac{v^2}{2} + w\sigma - e\right)_q = 0.$$

Variational principle

$$a = \int_{t_0}^{t_1} \int_{-\infty}^{+\infty} L\left(x_t, x_q\right) dt dq = \int_{t_0}^{t_1} \int_{-\infty}^{+\infty} \left(\frac{x_t^2}{2} - e\left(x_q\right) \right) dt dq$$

Euler-Lagrange equations

$$-\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_t}\right) - \frac{\partial}{\partial q}\left(\frac{\partial L}{\partial x_q}\right) = 0 \Leftrightarrow -\frac{\partial^2 x}{\partial t^2} + \frac{\partial}{\partial q}\left(\frac{\partial e}{\partial x_q}\right) = 0 \Leftrightarrow \frac{\partial v}{\partial t} - \frac{\partial \sigma}{\partial q} = 0$$

Generalized σ-systems

Lagrangian

$$L = \frac{x_t^2}{2} - e\left(x_q, x_{qt}, x_{qq}\right) = \frac{v^2}{2} - e\left(w, w_t, w_q\right)$$

Governing equations

$$w_t - v_q = 0,$$

$$v_t - \sigma_q = 0.$$

Expression for the stress

$$\sigma = \frac{\delta e}{\delta w} = \left(\frac{\partial e}{\partial w} - \frac{\partial}{\partial t}\left(\frac{\partial e}{\partial w_t} \right) - \frac{\partial}{\partial q}\left(\frac{\partial e}{\partial w_q} \right) \right)$$

Additional conservation laws (general case)

$$\left(E+\frac{v^2}{2}\right)_t-\left(\sigma v+v_q\frac{\partial e}{\partial w_q}\right)_q=0,$$

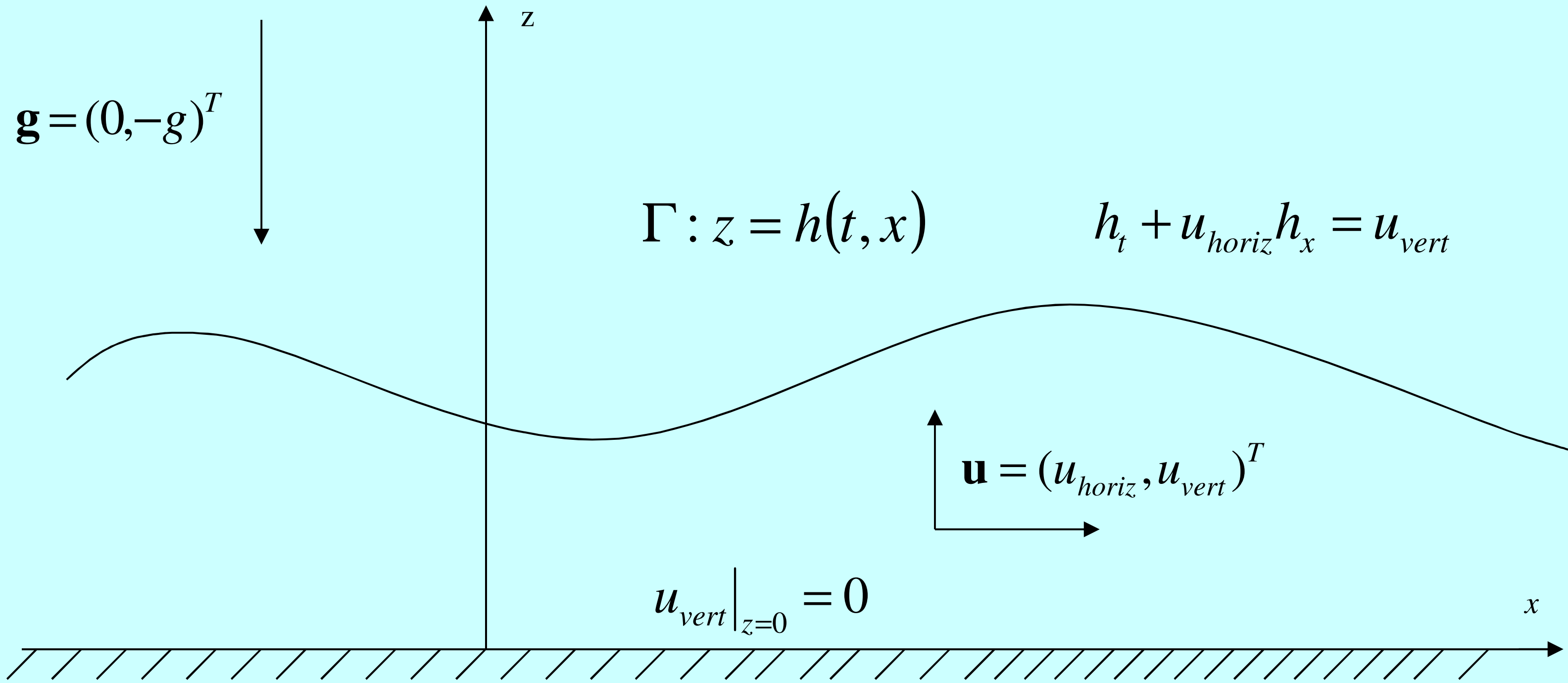
$$\left(wv+\tau w_q\right)_t-\left(\frac{v^2}{2}+w\sigma+w_q\frac{\partial e}{\partial w_q}-e\right)_q=0,$$

$$E=e+\tau w_t,\quad \tau=-\frac{\partial e}{\partial w_t}.$$

1. interstitial working terms

2. *e* is not the internal energy, but its Legendre transform *E* is!

Examples



z
$\mathbf{g} = (0, -g)^T$
$\Gamma : z = h(t,x)$
$h_t + u_{horiz} h_x = u_{vert}$
$\mathbf{u} = (u_{horiz}, u_{vert})^T$
$u_{vert}\big|_{z=0} = 0$
x

Examples (1)

Long water waves with surface tension effects
(V. Nikolayev, SG and H. Gouin, 2006)

$$\Lambda = \int_{-\infty}^{+\infty} \int_0^h \left(\frac{1}{2}\|\mathbf{u}\|^2 - gz \right) dx dz - \sigma_s \int_{-\infty}^{+\infty} \left(\sqrt{1+h_x^2} - 1 \right) dx$$

Constraint $\quad div\,\mathbf{u} = 0$

Approximate Lagrangian

$$\Lambda = \int_{-\infty}^{+\infty} \left(\frac{hv^2}{2} - \frac{gh^2}{2} - \frac{\sigma_s h_x^2}{2} \right) dx = \int_{-\infty}^{+\infty} \left(\frac{v^2}{2} - \frac{gh}{2} - \frac{\sigma_s h_x^2}{2h} \right) dq$$

$$v = \frac{1}{h}\int_0^h u_{horiz} dz, \quad h_t + (hv)_x = 0 \Leftrightarrow w_t - v_q = 0, \quad w = 1/h.$$

$$e = \frac{gh}{2} + \frac{\sigma_s h_x^2}{2h} = \frac{g}{2w} + \sigma_s \frac{w_q^2}{2w^7}, \quad h_x = h_q q_x = \frac{h_q}{x_q} = -\frac{w_q}{w^3}.$$

Examples (2)

Long water waves without surface tension
(Serre-Su-Gardner-Green-Naghdi model)

$$L=\int_{-\infty}^{+\infty}\int_{0}^{h}\left(\frac{1}{2}\|\mathbf{u}\|^2-gz\right)dxdz$$

Constraint $div\,\mathbf{u}=0$

Approximate Lagrangian (for almost potential flows)

$$L=\int_{-\infty}^{+\infty}\left(\frac{v^2}{2}+\frac{1}{6}\left(\frac{Dh}{Dt}\right)^2-\frac{gh}{2}\right)hdx,\quad \frac{Dh}{Dt}=h_t+vh_x.$$

$$e=\frac{gh}{2}-\frac{1}{6}\left(\frac{Dh}{Dt}\right)^2=\frac{g}{2w}-\frac{w_t^2}{6w^4}$$

Other fields of applications

- Van-der-Waals theory of capillarity
- Non-linear Schrödinger equation
- Two-temperature collisionless plasma
- Bubbly fluids
- …

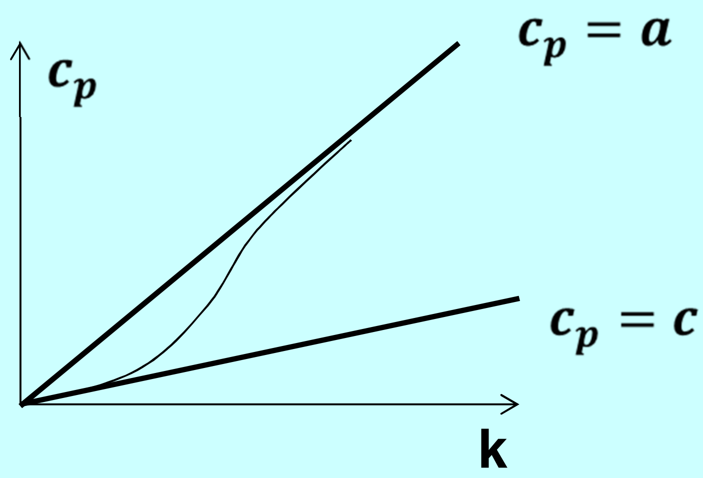
P. Casal, H. Gouin, M. Slemrod, S. Benzoni-Gavage, Ph. LeFloch, G. El, A. Kamchatnov, M. Hoefer, …

Simplified dispersion (general case)

$$e = e_0(w) + \frac{\mu^2}{2}\left(w_q^2 - \frac{w_t^2}{a^2} \right)$$

$$E = e_0(w) + \frac{\mu^2}{2}\left(w_q^2 + \frac{w_t^2}{a^2} \right)$$

Dispersion relation (c<a)



Whitham averaging

$$\langle f \rangle (t,q) = \int_0^1 f(\theta,t,q)\,d\theta$$

Generating function H

$$H\left(\omega,k,\langle\sigma\rangle\right)=\langle w\rangle\frac{\partial\langle L\rangle\left(\omega,k,\langle w\rangle\right)}{\partial\langle w\rangle}-\langle L\rangle\left(\omega,k,\langle w\rangle\right)$$

Modulation equations

$$\left(\frac{\partial H}{\partial\langle\sigma\rangle}\right)_t + \langle v\rangle_q = 0,$$

$$\langle v\rangle_t - \langle\sigma\rangle_q = 0,$$

$$\left(\frac{\partial H}{\partial\omega}\right)_t - \left(\frac{\partial H}{\partial k}\right)_q = 0,$$

$$k_t + \omega_q = 0.$$

Additional conservation laws

$$\left(\frac{\langle v\rangle^2}{2} + H - \omega H_\omega - \langle\sigma\rangle H_{\langle\sigma\rangle}\right)_t - \left(\langle v\rangle\langle\sigma\rangle - \omega H_k\right)_q = 0,$$

$$\left(\langle v\rangle H_{\langle\sigma\rangle} - kH_\omega\right)_t + \left(\frac{\langle v\rangle^2}{2} + kH_k - H\right)_q = 0.$$

A sufficient hyperbolicity criterion can be formulated in terms of the convexity of the function *H*.

$$A\mathbf{U}_t + B\mathbf{U}_x = 0, \quad A = A^T > 0, \quad B = B^T$$

Drawback : H is not explicit ! Idea : to find a simple explicit expression.

Particular case

$$H = F(\omega,k)\frac{\langle\sigma\rangle^2}{2}$$

Characteristic polynomial

$$F^3\left(\left(\frac{1}{F}\right)_{\omega\omega}\lambda^2+2\left(\frac{1}{F}\right)_{\omega k}\lambda+\left(\frac{1}{F}\right)_{kk}\right)\lambda^2-\left(F_{\omega\omega}\lambda^2+2F_{\omega k}\lambda+F_{kk}\right)=0$$

Lemma 4 *Let ω and k be such that there exists a real number s satisfying the following relations*

$$s^2=-F^{-1}(\omega,k)>0, \quad s=-\frac{\left(F^{-1}\right)_k}{\left(F^{-1}\right)_\omega}\equiv-\frac{F_k}{F_\omega} \tag{39}$$

Here we suppose that $F_\omega \neq 0$. Then $\lambda = s$ is a characteristic eigenvalue i.e. it verifies (38) (or (37)).

Moreover, if

$$s^2F_{\omega\omega}+2sF_{\omega k}+F_{kk}\neq 0 \tag{40}$$

the relation (39) defines a curve. Condition (39) implies, in particular, that

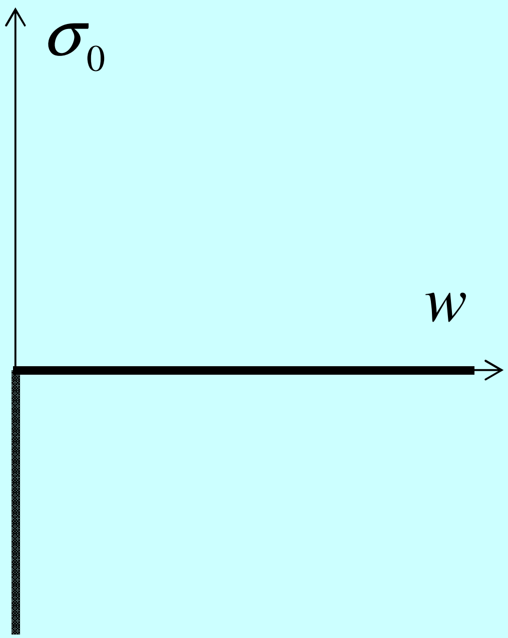
$$\lambda_k+\lambda\lambda_\omega\neq 0 \tag{41}$$

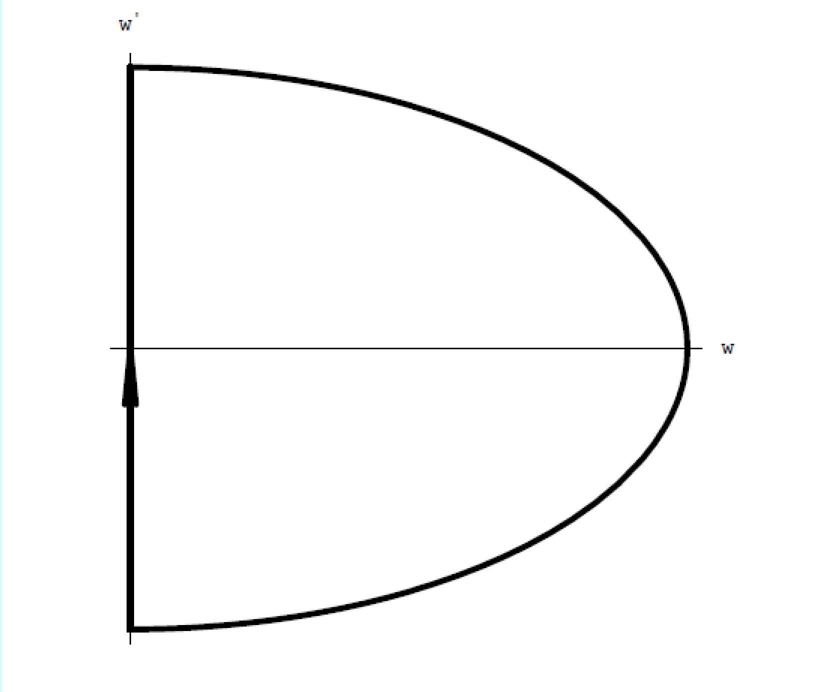
along this curve.

Condition for genuinely non-linear fields

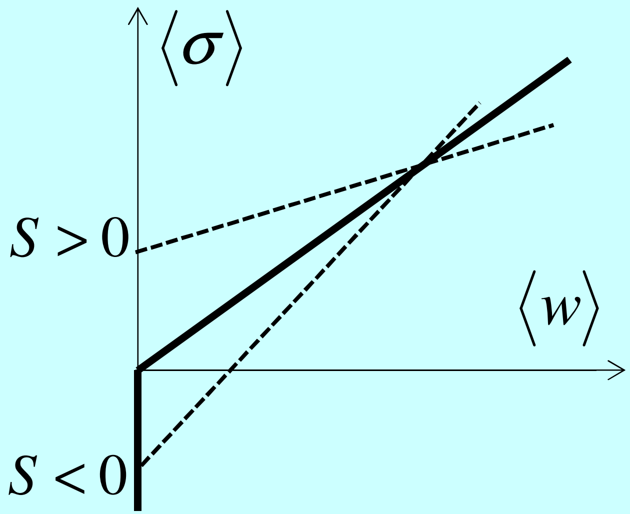
$$\nabla \lambda_i \cdot \mathbf{r}_i = \lambda_i \frac{\partial \lambda_i}{\partial \omega} + \frac{\partial \lambda_i}{\partial k} \neq 0$$

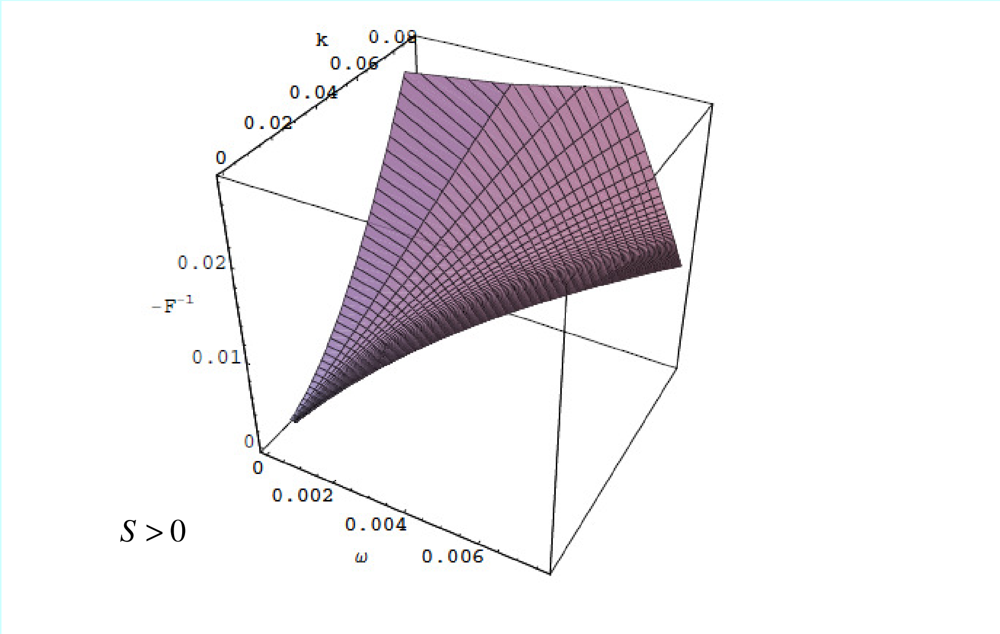
`hard spheres' case ($c = 0, \quad w \geq 0.$)





Effective Young modulus $-F^{-1}$





A good 'Cartesian' parametric form of $-F^{-1}$ is obtained!

$$S>0\,(S<0)\Leftrightarrow\sqrt{-F^{-1}}>\frac{\omega}{k}\left(\sqrt{-F^{-1}}<\frac{\omega}{k}\right)$$

Hyperbolicity property

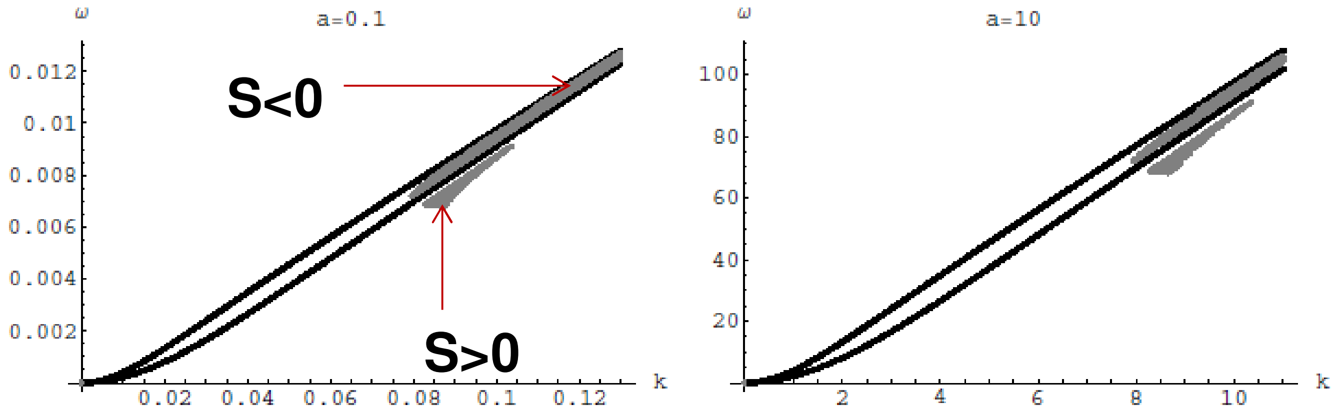


Figure 8: For any a the hyperbolicity domain (shown in grey color) is always compact for $S > 0$ (the domain which is at the right of the bold curve defined by (56), and semi-infinite in the case $S < 0$ (the domain which is between the bold curves defined by (57)). The domains of hyperbolicity move to higher frequences and higher wave numbers when the parameter a increases (the inertia type of regularization becomes less important than the capillarity type of regularization). Moreover, the compact domain of hyperbolicity (for $S > 0$) increases in size when a increases (see the scales of k and ω). We have shown this evolution for $a = 0.1$ and $a = 10$.

One negative and three positive eigenvalues in the hyperbolicity region

Rankine-Hugoniot relations for the `hard spheres' system

$$\left(\langle\sigma\rangle F\right)_t + \langle v\rangle_q = 0,$$

$$\langle v\rangle_t - \langle\sigma\rangle_q = 0,$$

$$\left(\frac{\langle v\rangle^2}{2} + \frac{\langle\sigma\rangle^2}{2}\left(F - \omega F_\omega\right)\right)_t - \left(\langle v\rangle\langle\sigma\rangle - \frac{\langle\sigma\rangle^2}{2}\omega F_k\right)_q = 0.$$

What is the entropy?

$$\left(\frac{\langle\sigma\rangle^2}{2}\left(F\right)_\omega\right)_t - \left(\frac{\langle\sigma\rangle^2}{2}\left(F\right)_k\right)_q = 0?$$

$$\left(\langle v\rangle\langle\sigma\rangle F - k\frac{\langle\sigma\rangle^2}{2}F_\omega\right)_t + \left(\frac{\langle v\rangle^2}{2} + \frac{\langle\sigma\rangle^2}{2}\left(kF_k - F\right)\right)_q = 0?$$

$$k_t + \omega_q = 0?$$

Singular solution for the `hard spheres' system

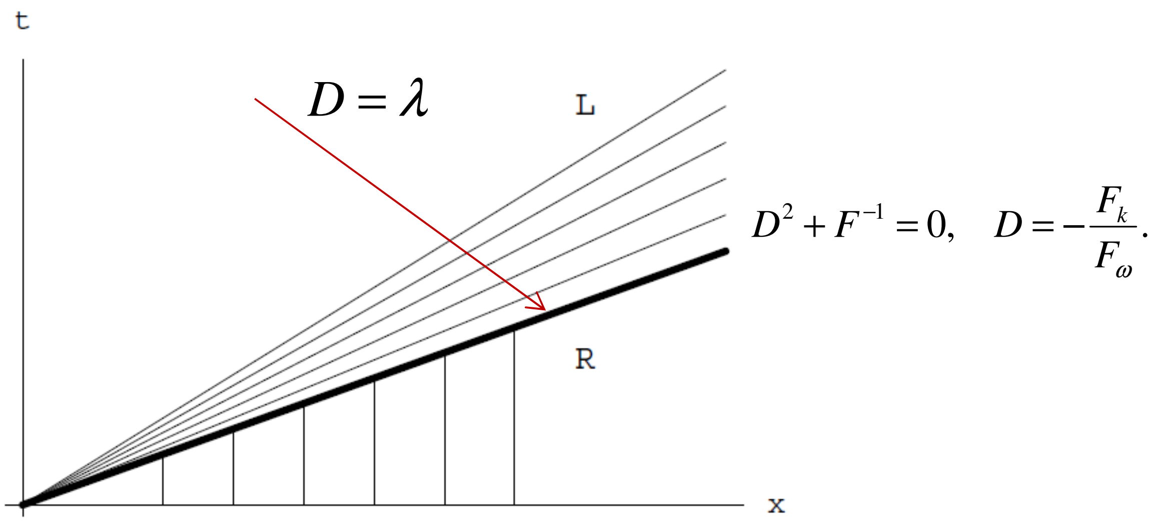
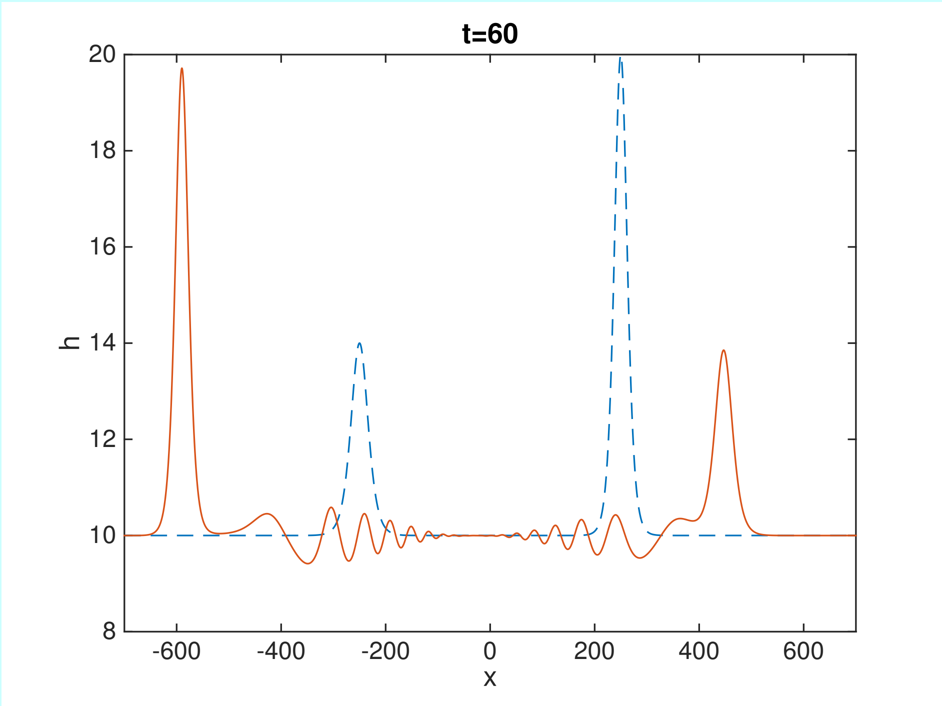


Figure 19: A solution relating the immobile state R ($\omega = 0, k = 0$, $\langle\sigma\rangle = 0$, $\langle w \rangle = 0$) with a state L by a characteristic shock following by a rarefaction fan.

For such a singular solution all conservation laws are satisfied excepting the conservation law for the phase.

Conclusions and perspectives

1. The inertia `stabilizes better' the modulation equations (at least in `supersonic' part of the domain).
2. Efficient numerical schemes for dispersive equations are needed to understand better interaction phenomena.



Keh-Ming Shyue (2015, work in progress …)