The reduced Ostrovsky equation: integrability and breaking

Ted Johnson (UCL),
Roger Grimshaw (UCL),
Karl Helfrich (WHOI)

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The reduced Ostrovsky equation

KdV with weak rotation: Ostrovsky equation

$$u_t + \mu u u_x + \lambda u_{xxx} = \gamma v, \qquad v_x = \gamma u.$$

- μ nonlinearity; λ non-hydrostatic; γ rotation
- $\lambda = 0$ and $\gamma = 0$ (non-rotating, hydrostatic)
  Inviscid Burgers (Hopf) equation
  All localised or periodic solutions break
- $\gamma = 0$ and $\lambda \neq 0$ (non-rotating, non-hydrostatic): KdV
  No regular initial conditions break
- $\lambda = 0$ and $\gamma \neq 0$ (rotating, hydrostatic)
  Reduced Ostrovsky (Hunter, Vakhnenko) equation.
  Some initial conditions break, others do not

The reduced Ostrovsky equation

Rescale equation ($\mu = 1$, $\gamma = 1$). Introduce anti-differentiation operator for localised or periodic initial data

$$\partial_x^{-1} u = \int^x u(x', t)\mathrm{d}x',$$

with integration constant chosen so integral over domain or period vanishes (to satisfy zero-mass constraint). Then

$$u_t + uu_x = \partial_x^{-1} u, \qquad (1)$$

the reduced Ostrovsky equation.

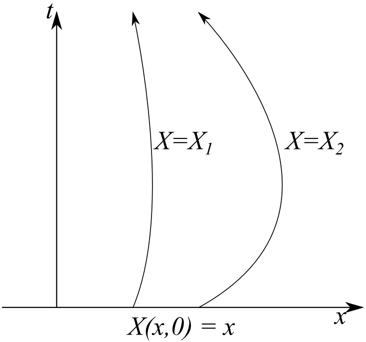
Previous work

Hunter (1990)
Vakhnenko (1992)
Parkes (1993)
Vakhnenko and Parkes (1998)
Boyd (2004, 2005) (microbreaking)
Stepanyants (2006)
Esler, Rump & Johnson (2009)
Liu *et al* (2010)
Kraenkel *et al* (2011)

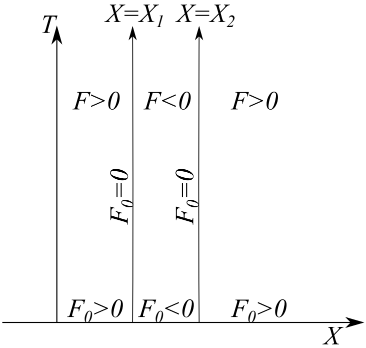
Characteristic co-ordinates

- The RedO
$$u_t + uu_x = \partial_x^{-1} u, \qquad (2)$$
is a quasi-linear first-order pde with one set of characteristics.
- On characteristics
$$\frac{\mathrm{d}x}{\mathrm{d}t} = u, \qquad \frac{\mathrm{d}u}{\mathrm{d}t} = \partial_x^{-1} u.$$
- Let the characteristics be the lines $\mathcal{X}(x, t) = \mathrm{constant}$. Lagrangian co-ordinate (Zeitlin *et al.* 2003, 1D rSWE).
- In terms of $(\mathcal{X}, T)$ with $t = T$ and $u(x, t) = U(\mathcal{X}, T)$
$$x_T = U, \qquad U_T = \partial_x^{-1} U,$$
with $\mathcal{X} = x$ at $T = 0$.

Characteristic co-ordinates



Laboratory frame



Characteristic (Lagrangian) frame

Characteristic co-ordinates

Our system is thus

$$x_T = U, \qquad U_T = \partial_x^{-1} U,$$

with $\mathcal{X} = x$ at $T = 0$.

Differentiating wrt $\mathcal{X}$ gives the pair

$$x_{\mathcal{X}T} = U_{\mathcal{X}}, \qquad U_{\mathcal{X}T} = \partial_{\mathcal{X}} \partial_x^{-1} U = x_{\mathcal{X}} \partial_x \partial_x^{-1} U = x_{\mathcal{X}} U,$$

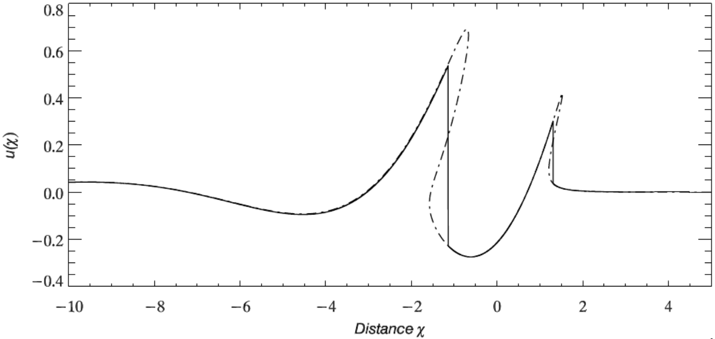
i.e.

$$\phi_T = W, \qquad W_T = \phi U,$$

where $W = U_{\mathcal{X}}$ and $\phi = x_{\mathcal{X}}$ is the Jacobian of the transformation to characteristic co-ordinates.

The Jacobian, ϕ

- ϕ is initially unity
- Provided ϕ remains bounded and positive the transformation is 1:1 and the wave does not break.
- If ϕ passes through zero then the waves overturns (breaks). (Nothing untoward numerically).



Esler, Rump & Johnson (2009).

Kraenkel *et al* (2011)

Differentiating (1) w.r.t. x twice and rearranging gives

$$F_t + (uF)_x = 0 \, .$$

where

$$F^3 = 1 - 3u_{xx} \, .$$

i.e. F is a conserved density.

The density $F = (1 - 3u_{xx})^{1/3}$ in characteristic co-ordinates

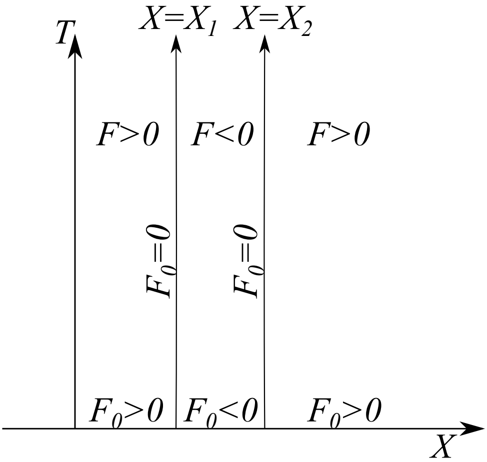
$$(F\phi)_T = 0\,, \quad \text{so that} \quad F\phi = F_0(\mathcal{X}),$$

$$\text{where} \qquad F_0(\mathcal{X}) = F(\mathcal{X}, 0) = F(x, 0),$$

determined by the initial conditions.

- Until breaking $\phi > 0$. Thus $F(\mathcal{X}, T) = F_0(\mathcal{X})/\phi(\mathcal{X}, T)$
- On each characteristic
  - If $F_0(\mathcal{X}) > 0$, then $F(\mathcal{X}, T) > 0, \quad \forall T \geq 0$.
  - If $F_0(\mathcal{X}) < 0$, then $F(\mathcal{X}, T) < 0, \quad \forall T \geq 0$.
  - If $F_0(\mathcal{X}) = 0$, then $F(\mathcal{X}, T) = 0, \quad \forall T \geq 0$.
- Until breaking, the $\mathcal{X}$-domain is permanently divided by the initial conditions into $\mathcal{X}$-intervals where $F > 0$ and the remaining $\mathcal{X}$-intervals where $F < 0$.

The density F in characteristic co-ordinates



Characteristic (Lagrangian) frame

Reduction of order, $F = (1 - 3u_{xx})^{1/3}$

Now $$u_{xx} = \frac{1}{\phi}\{\frac{U_{\mathcal{X}}}{\phi}\}_{\mathcal{X}} = \frac{1}{\phi}\{\frac{\phi_T}{\phi}\}_{\mathcal{X}} = \frac{\{\log\phi\}_{\mathcal{X}T}}{\phi},$$

i.e. $$F^3 = 1 - (3/\phi)\{\log\phi\}_{\mathcal{X}T}.$$

Combining this with $F\phi = F_0(\mathcal{X})$ gives

$$(\log\phi)_{\mathcal{X}T} = \frac{\phi}{3}(1 - \frac{F_0^3}{\phi^3}), \tag{3}$$

or $$(\log F)_{\mathcal{X}T} = \frac{F_0}{3F}(F^3 - 1), \tag{4}$$

equations for ϕ and F alone.

Integrability: $F_0(\mathcal{X}) > 0 \ \forall \mathcal{X}$, $\qquad F = (1 - 3u_{xx})^{1/3}$

following Kraenkel *et al.*(2011)

- For smooth bounded initial conditions $u_{xx} = 0$ somewhere.
- Thus $F_0(\mathcal{X}) = 1$ for some $\mathcal{X}$.
- Thus suppose $F_0(\mathcal{X}) > 0 \ \forall \mathcal{X}$ at $T = 0$.
- Introduce ζ through the 1:1 mapping defined by

  $$\mathrm{d}\zeta = (1/3) F_0(\mathcal{X})\, \mathrm{d}\mathcal{X}.$$

- Then equations (3),(4) reduce to the *integrable* Tzitzeica (1910) equation

  $$(\log h)_{\zeta T} = h - h^{-2},$$

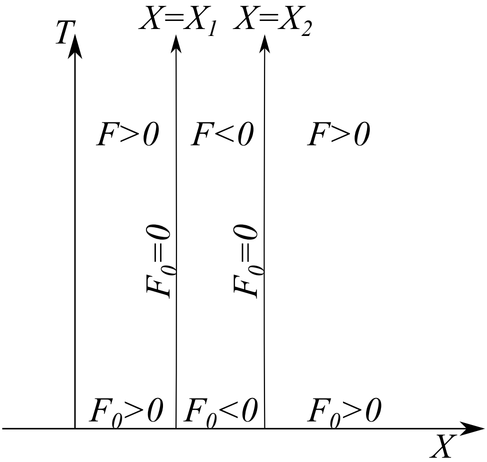
  where $h = \phi/F_0 = 1/F$. (Kraenkel *et al.* : Dodd-Bullough, 1977, equation)
- Rigorous: Grimshaw & Pelinovsky (2014)

Integrability

- Hence the RedO (1) is integrable for initial data such that $F_0 > 0$
- i.e. if $u_{xx} < 1/3$ everywhere at any instant (including $t = 0$), then the interface evolves for all time without breaking (and such that $u_{xx} < 1/3$ everywhere)
- This remains true even if $F_0(\mathcal{X})$ vanishes at isolated values of $\mathcal{X}$ (since the transformation to ζ remains 1:1).
- Now suppose there exists an interval $x_1 \leq x \leq x_2$ in which $u_{0xx} \geq 1/3$, with equality only at the end points. Then $F_0(x) \leq 0$ so

$$F(\mathcal{X}, T) < 0, \quad \forall \mathcal{X}_1 < \mathcal{X} < \mathcal{X}_2, \quad \forall T \geq 0.$$

F negative in an interval



Characteristic (Lagrangian) frame

The interval $\mathcal{X}_1 < \mathcal{X} < \mathcal{X}_2$, $F_0(\mathcal{X}) < 0$

- Integrating equation (3) for ϕ in time (i.e. wrt T) gives

$$\beta(\mathcal{X}, T) = (\log\phi)_{\mathcal{X}} = \int_0^T \frac{\phi}{3}\big(1 - \frac{F_0^3}{\phi^3}\big)\,dT\,. \qquad (5)$$

- The integrand is positive for all $\phi > 0$, with a minimum value of $-2^{-2/3}F_0(\mathcal{X})$ achieved where $\phi = -2^{1/3}F_0(\mathcal{X})$, independently of T.
- Thus $\beta > 0$ in $\mathcal{X}_1 \leq \mathcal{X} \leq \mathcal{X}_2$. So $\phi_{\mathcal{X}} > 0$ there. So ϕ cannot achieve a minimum value in this interval.
- Thus breaking (if it occurs) occurs first at a point corresponding to $u_{xx} < 1/3$ in initial data (the integrable region).

Breaking

- Now, for each $\mathcal{X}$ in the interval $\mathcal{X}_1 < \mathcal{X} < \mathcal{X}_2$,

$$\beta = (\log \phi)_{\mathcal{X}} > -2^{-2/3} F_0(\mathcal{X}) T \,,$$

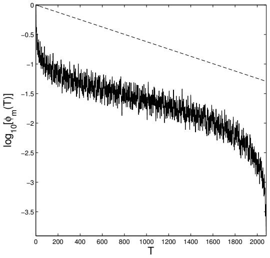
- Integrating over the interval $\mathcal{X}_1 < \mathcal{X} < \mathcal{X}_2$ yields

$$\phi(\mathcal{X}_1, T) < \phi(\mathcal{X}_2, T) \exp\left(-\alpha T\right),$$

$$\alpha = 2^{-2/3} \int_{\mathcal{X}_1}^{\mathcal{X}_2} (-F_0(\mathcal{X}))\, d\mathcal{X} = 2^{-2/3} \int_{x_1}^{x_2} \{3u_{0xx}(x) - 1\}^{1/3}\, dx \,.$$

- Thus the Jacobian $\phi(\mathcal{X}_1, T)$ at the left-hand end of the interval on which F_0 is negative becomes exponentially small compared to its value $\phi(\mathcal{X}_2, T)$ at the right-hand end.

Jacobian minimum

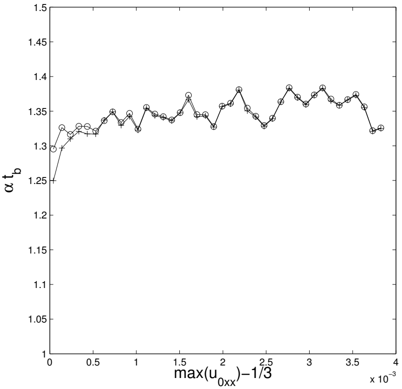


The logarithm of the minimum, $\phi_m(T)$, over $\mathcal{X}$ of the Jacobian $\phi(\mathcal{X}, T)$ as a function of T for the initial profile

$$u_0(x) = u_1 \sin(x) + u_2 \sin(2x + \theta),$$

(where θ is an arbitrary phase shift). Here $u_1 = 0.3$, $u_2 = 0.03$ and $\theta_0 = 3.5453$ so $\max(u_{0xx}) - 1/3 = 4 \times 10^{-5}$, computed with $N = 4096$ nodes. The wave breaks when ϕ_m first vanishes, at $T = t_b = 2081.7$.

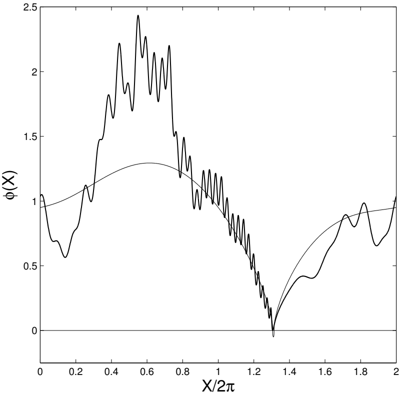
Breaking-time scaling



The scaled time to breaking, αt_b, for this initial profile for varying θ_0 as a function of the excess of u_{0xx} over $1/3$.

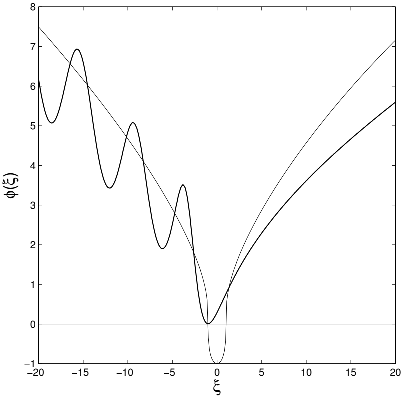
The number of nodes in the computations are: '$+$' $N = 2048$ and 'o' $N = 4096$.

Jacobian at breaking



The Jacobian $\phi(\mathcal{X}, t_b)$ at the instant of breaking. The thinner curve shows $F_0(\mathcal{X})$ which is negative in a region surrounding 1.31π.

Jacobian at breaking - detail



The scaled Jacobian $\phi(\hat{\xi})$ as a function of the scaled co-ordinate ξ. The scaling is such that the region of negative $F_0(\mathcal{X})$ has unit depth and width 2.

Jacobian at breaking - asymptotic form

- Consider a weakly supercritical initial condition where u_{0xx} is smooth with maximum at $\mathcal{X}_0$ slightly exceeding 1/3.
- Near $\mathcal{X}_0$,

$$u_{0xx} = a - b(\mathcal{X} - \mathcal{X}_0)^2 + \cdots ,$$

  where $a = \max(u_{0xx}) = u_{0xx}(\mathcal{X}_0)$ and $b = -(1/2)u_{0xxxx}(\mathcal{X}_0) > 0$.

- Then

$$[F_0(\mathcal{X})]^3 = (3a-1)[-1+\xi^2+\cdots],$$

  where $\xi = (\mathcal{X} - \mathcal{X}_0)[3b/(3a-1)]^{1/2}$ and $\xi = \pm 1$ corresponds to $\mathcal{X} = \mathcal{X}_2, \mathcal{X}_1$ in the general problem.

- Write

$$\phi = (3a-1)^{1/3}\hat{\phi},$$

  giving the *parameter-free* generic equation near breaking,

$$(\log \hat{\phi})_{\xi\mathcal{T}} = (\hat{\phi}/3)[1 + (1-\xi^2)/\hat{\phi}^3],$$

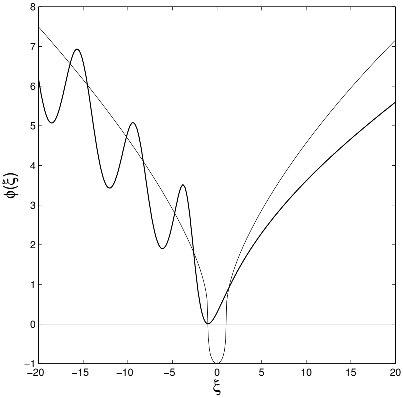
  where $T = \epsilon\tau$ for $\epsilon = (3a-1)^{5/6}/\sqrt{3b}$.

- *The time to breaking scales as* $[\max(u_{0xx}) - 1/3]^{5/6}$.

Jacobian minimum at large time

- Dropping the first term in the governing equation (less than 1/8th the second) gives
$$(\log\phi)_{\mathcal{X}T} = -(1/3)F_0^3/\phi^2.$$
- This has solution
$$\phi = A + B(\mathcal{X})T,$$
for A constant and $B(\mathcal{X})$ a function of $\mathcal{X}$ alone, provided $AB'(\mathcal{X}) = -(1/3)F_0^3$.
- Near breaking
$$\phi = A(1 - t/t_b) + (t/3A)\int_{\mathcal{X}}^{\mathcal{X}_1} F_0^3(\mathcal{X}')\mathrm{d}\mathcal{X}'.$$
- Since $F_0 > 0$ in $\mathcal{X} < \mathcal{X}_1$ and $F_0 < 0$ in $\mathcal{X} > \mathcal{X}_1$ this gives ϕ increasing monotonically with distance from a local minimum at $\mathcal{X} = \mathcal{X}_1$ of
$$\phi_m = A(1 - t/t_b).$$
- The Jacobian does indeed appear to decrease linearly with t at large t until vanishing at t_b.

Jacobian at breaking - detail



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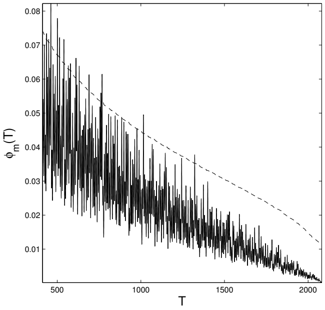
Jacobian minimum at large time



The minimum of the Jacobian, $\phi_m(T)$, as a function of time for $T > 400$. The dashed line shows the corresponding value of F_0 at the same $\mathcal{X}$ and T, i.e. $F_m(T) = F_0(\mathcal{X}_m(T))$. Note that at large T, ϕ_m is less than $\frac{1}{2}F_m$.

Ostrovsky number

- In the unscaled equation an Ostrovsky number can be defined as

$$O_s = 3\mu\kappa/\gamma^2\,, \quad \text{where} \quad \kappa = \max[u_{0xx}(x)]\,.$$

- Initial conditions with $O_s > 1$ break and those with $O_s \leq 1$ do not.
- Increasing nonlinearity (μ) or curvature (κ) increases O_s.
- Increasing rotation (γ) decreases O_s.

Modified reduced Ostrovsky equation

- A rotating, hydrostatic, two-layer, Boussinesq fluid where the layers have equal depths, is governed by the mRO

$$u_t + (1/2)u^2 u_x = \partial_x^{-1} u.$$

- Similar considerations show that
  - If $|u_{0x}| < 1$ everywhere, the wave never breaks.
  - If $|u_{0x}| > 1$ somewhere, the wave breaks in finite time.

Grimshaw, Helfrich, and Johnson, Stud. Appl. Math., **129**, 414-436, (2012).

Johnson and Grimshaw, Physical Review E, **88**, (2013).