THE MATHEMATICAL THEORY OF SMALL-SCALE DEPENDENT SHOCK WAVES

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An example: the singular limit $\varepsilon, \kappa \to 0$

$$\rho_t + (\rho u)_x = 0$$
$$(\rho u)_t + \left(\rho u^2 + k\,\rho^2\right)_x = \varepsilon\, u_{xx} + \kappa\,(\rho^2 \rho_{xx})_x$$

- $\varepsilon = \kappa = 0$: Euler system (shock formation, weak solutions)
- $\varepsilon^2 \simeq \kappa \to 0$: singular limit problem (diffusive-dispersive regime)
- $\kappa = \alpha\,\varepsilon^2 \to 0$: (vanishing) diffusive-dispersive shock waves *depending upon* α !

Conservation laws with vanishing diffusion, dispersion, etc.

$$u_t^\varepsilon + f(u^\varepsilon)_x = R(\varepsilon u_x^\varepsilon, \varepsilon^2 u_{xx}^\varepsilon, \dots)_x$$

- $u = \lim_{\varepsilon \to 0} u_\varepsilon$: shock wave solutions to $u_t + f(u)_x = 0$
- Second-order: $\varepsilon\, u_{xx}^\varepsilon$ (viscosity). Lax's theory of shock waves (entropy condition, compressive shocks)
- Third- or higher-order: $\alpha\, \varepsilon^2\, u_{xxx}^\varepsilon$ (capillarity)
  Oscillations near shocks, driven by dispersive effects, delicate competition between "small scales"
- Classical compressive + nonclassical undercompressive shocks (or subsonic phase boundaries).

The mathematical theory

- Dynamics of diffusive-dispersive shocks, nonlinear interactions
- Internal shock structure, analysis of diffusive-dispersive traveling waves
- Develop general mathematical methods for these singular limit problems
- Design numerical methods adapted to small-scale dependent shocks

THE MATHEMATICAL THEORY OF NONCLASSICAL SHOCK WAVES

1. **Diffusive-dispersive models**
   (non-convexity, entropy inequality)
2. **Nonclassical Riemann solver** with entropy-compatible kinetics
   (general theory, single jump initial data, kinetic function)
3. **Kinetic functions based on traveling waves**
   (shock structure, specific models)
4. **The initial value problem** for arbitrary initial data
   (finite total variation, $u_x \in \mathcal{M}$, generalized TV functional, Glimm method)
5. **Schemes with well-controled dissipation** (WCD)
   (entropy conservative discrete flux, equivalent equation)
6. **Computing kinetic functions**
   (effect of the parameter α)
7. **The zero diffusion-dispersion limit**
   (finite energy: $u \in L^2$, weak convergence, conserved quantities)

First developments. Materials undergoing phase transitions

$$w_t - v_x = 0$$
$$v_t - \sigma(w)_x = \varepsilon\, v_{xx} - \alpha\, \varepsilon^2\, w_{xxx}$$

v : velocity $\quad w > -1$: deformation gradient $\quad \sigma(w)$: stress

ε : viscosity $\quad \alpha\, \varepsilon^2$: capillarity

- Slemrod (1984, etc): self-similar solutions
- Shearer (1986, etc.): Riemann problem
- Truskinovsky (1987, etc): kinetic relation
- Abeyaratne & Knowles (1990, etc): trilinear equation, nucleation

- *PLF, Propagating phase boundaries. Formulation of the problem and existence via the Glimm scheme, Arch. Rational Mech. Anal. 123 (1993)*
  - formulation for general hyperbolic systems
  - weak solutions with finite total variation
  - Cauchy problem and the Glimm method

1. DIFFUSIVE-DISPERSIVE MODELS

Vanishing linear diffusion-dispersion.

- Conservation law $$u_t + f(u)_x = \varepsilon\, u_{xx} + \kappa\, u_{xxx}$$

  (Shearer et al., Hayes-PLF, LeFloch, Bedjaoui-PLF)

- Classical/nonclassical solutions
  - $\kappa << \varepsilon^2$ (dominant diffusion)
    (Lax, classical) entropy solutions
  - $\kappa >> \varepsilon^2$ (dominant dispersion)
    high oscillations, weak convergence (Lax, Levermore)
  - $\kappa = \alpha\, \varepsilon^2$ (balanced regime)
    strong convergence, mild oscillations, nonclassical, depend on α

- Entropy inequality with entropy flux $F'(u) = uf'(u)$

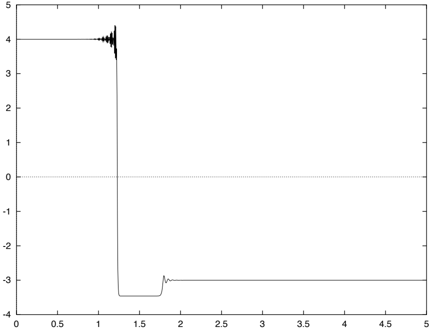
$$\frac{1}{2}\big(u^2\big)_t + \big(F(u)\big)_x = -D + C_x, \qquad D = \varepsilon\, |u_x|^2 \geq 0$$

$$C = \varepsilon\, uu_x + \kappa\Big(u\, u_{xx} - (1/2)u_x^2\Big)$$

In the limit $\varepsilon, \kappa \to 0$, we obtain: $$\big(u^2/2\big)_t + F(u)_x \leq 0$$

Nonclassical wave patterns

For instance for $u_t + (u^3)_x = 0$



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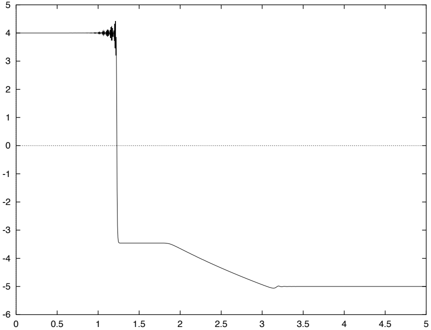


two shocks

shock + rarefaction

Solutions that are distinct from the ones selected by the standard (Oleinik) entropy conditions

Generalized Camassa-Holm model.

- Conservation law (with $\beta > 0$)

$$u_t + f(u)_x = \varepsilon\, u_{xx} + \kappa \Big(u_{txx} + 2u_x\, u_{xx} + u\, u_{xxx} \Big)$$

Shallow water model for wave breaking

- Limiting solutions when $\kappa = \alpha\, \varepsilon^2$: solutions are similar to, but do not coincide with, the ones obtained with the linear diffusion-dispersion model.

Weak solutions to $u_t + f(u)_x$ depend on the underlying small-scale physics

- Entropy inequality:

$$\frac{1}{2}\Big(u^2 + \kappa\, |u_x|^2\Big)_t + F(u)_x = -\, \varepsilon\, |u_x|^2 + C_x$$

In the limit $\varepsilon, \kappa \to 0$, we obtain again $\frac{1}{2}\Big(u^2\Big)_t + F(u)_x \le 0$

The inequality $\Big(u^2/2\Big)_t + F(u)_x \le 0$ is insufficient in order to formulate a suitable theory of (nonclassical) weak solutions to the hyperbolic conservation law.

Van der Waals fluids.

- Two coupled conservation laws

$$v_t - u_x = 0$$

$$u_t + p(v)_x = \Big(\varepsilon(v)\, u_x\Big)_x + \Big(\kappa'(v)\, \frac{v_x^2}{2} - \big(\kappa(v)\, v_x\big)_x\Big)_x$$

pressure law $p(v,T) = \frac{RT}{v-b} - \frac{a}{v^2}$

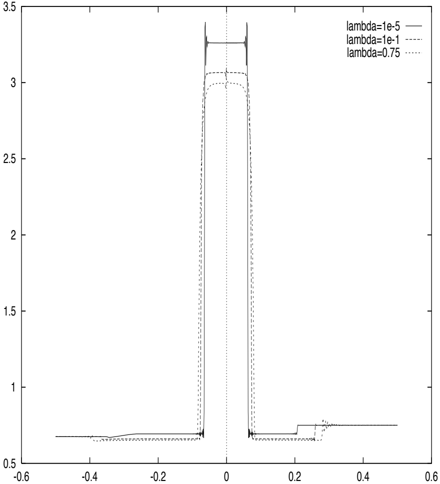
- Entropy inequality

$$\Big(e(v) + \frac{u^2}{2} + \kappa(v)\, \frac{v_x^2}{2}\Big)_t + \Big(p(v)\, u\Big)_x = -\, \varepsilon(v)\, u_x^2 + C_x$$

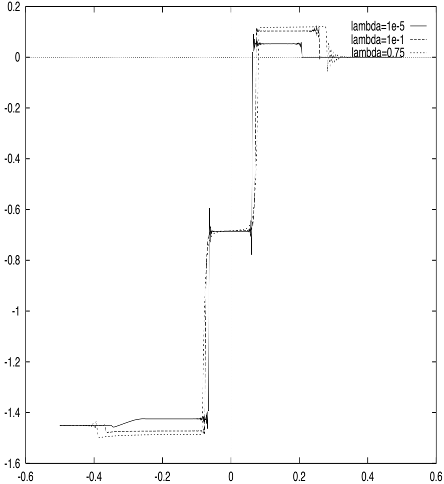
$e = e(v)$ internal energy.

In the limit $\varepsilon(v), \kappa(v) \to 0$, we obtain $\Big(e(v) + \frac{u^2}{2}\Big)_t + \Big(p(v)\, u\Big)_x \leq 0$

Nonclassical behavior for Van der Waals fluids



Specific volume v ,



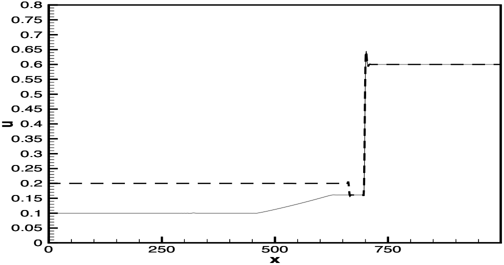
Velocity u

Weak solutions depend upon the ratio $\lambda = (\text{viscosity})^2/\text{capillarity}$

Thin liquid film model. (Bertozzi, Shearer, Münch, Levy)

- Conservation law (with $\varepsilon, \kappa > 0$)

$$u_t + (u^2 - u^3)_x = \varepsilon(u^3 u_x)_x - \kappa\,(u^3\, u_{xxx})_x$$



- Entropy inequality

$$\left(u \log u - u\right)_t + F(u)_x = -D + C_x$$

$$D = \varepsilon u^3 u_x^2 + \kappa\,|(u^2\, u_x)_x|^2 \geq 0$$

In the limit $\varepsilon, \kappa \to 0$, we obtain $\left(u \log u - u\right)_t + F(u)_x \leq 0$

Ideal magnetohydrodynamics with Hall effect.

- (v, w): transverse components of the magnetic field

$$v_t + ((v^2 + w^2)\, v)_x = \varepsilon\, v_{xx} + \alpha\epsilon\, w_{xx}$$
$$w_t + ((v^2 + w^2)\, w)_x = \varepsilon\, w_{xx} - \alpha\epsilon\, v_{xx}$$

ε: magnetic resistivity, α: Hall parameter

- Entropy inequality

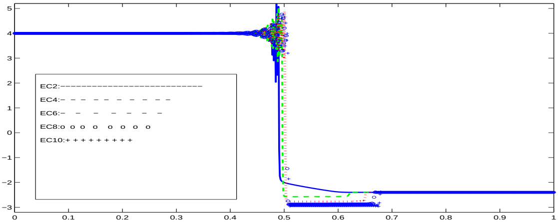
$$(1/2)\left(v^2 + w^2\right)_t + (3/4)\left((v^2 + w^2)^2\right)_x = -\,\varepsilon\,(v_x^2 + w_x^2) + C_x$$

- $\alpha = 0$: classical behavior

Brio, Hunter, Freistühler, Pitman, Panov, Wu, Kennel

- $\alpha \neq 0$: nonclassical behavior (plot of $r = (v^2 + w^2)^{1/2}$)

PLF-Mishra



FOR ALL THESE MODELS

- Complex wave patterns
- Different ratio/regularizations/schemes yield different solutions
- Non-convex flux-function and a single entropy inequality

THE MATHEMATICAL THEORY OF SMALL-SCALE DEPENDENT SHOCKS

- Include macro-scale effects without resolving the small-scales
- No "universal" admissibility criterion, but rather "several hyperbolic theories"
- Each being determined by specifying a physical regularization

$\longrightarrow$ KINETIC RELATION for undercompressive shocks (Truskinovsky, Abeyaratne-Knowles, PLF, Shearer, etc.)

$\longrightarrow$ DLM FAMILY of PATHS for nonconservative hyperbolic systems $u_t + A(u)u_x = 0$ (Dal Maso-LeFloch-Murat)

$\longrightarrow$ ADMISSIBLE BOUNDARY SETS (PLF-Dubois, PLF-Joseph, Serre) for the boundary value problem for hyperbolic problems

2. THE NONCLASSICAL RIEMANN SOLVER

$$u_t + f(u)_x = 0$$

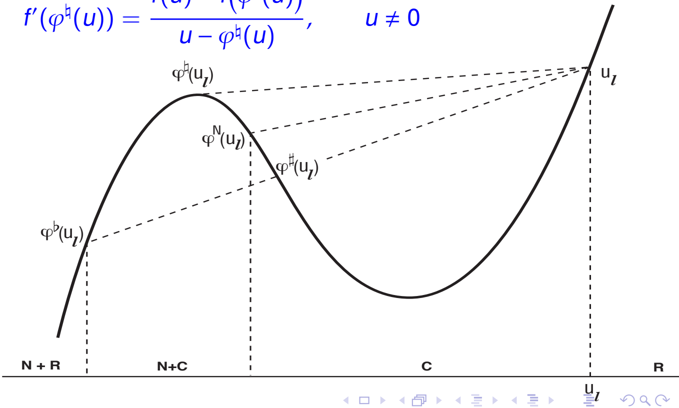
- Concave-convex flux

$$u\,f''(u) > 0 \quad (\text{for } u \neq 0)$$

$$f'''(0) \neq 0, \qquad \lim_{u \to \pm\infty} f'(u) = +\infty$$

- Tangent function $\varphi^\natural : \mathbb{R} \to \mathbb{R}$ and its inverse $\varphi^{-\natural}$

$$f'(\varphi^\natural(u)) = \frac{f(u) - f\big(\varphi^\natural(u)\big)}{u - \varphi^\natural(u)}, \qquad u \neq 0$$



Shock wave solutions.

$$u(t,x) = \begin{cases} u_-, & x < \lambda t \\ u_+, & x > \lambda t \end{cases}$$

satisfying the Rankine-Hugoniot relation $\lambda = \frac{f(u_-)-f(u_+)}{u_- - u_+} = \overline{a}(u_-, u_+)$

Standard Riemann solver based on the Oleinik entropy inequalities for shocks.

$$\frac{f(v) - f(u_+)}{v - u_+} \leq \frac{f(u_+) - f(u_-)}{u_+ - u_-}$$

for all v between u_- and u_+. Equivalent to imposing all of the entropy inequalities

$$U(u)_t + F(u)_x \leq 0$$

$$U'' > 0, \qquad F'(u) = f'(u)\,U'(u)$$

This condition characterizes *shock generated by diffusion only*

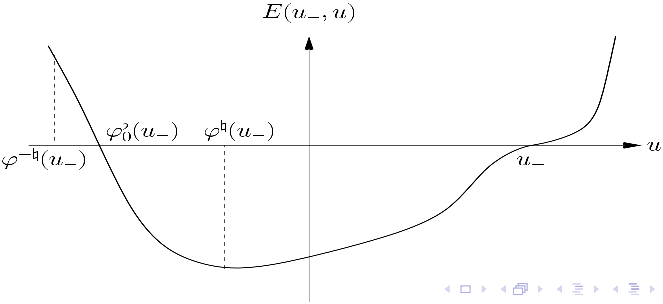
A single entropy inequality. This yields a much weaker condition

$$U(u)_t + F(u)_x \leq 0, \qquad U'' > 0, \qquad F'(u) = f'(u)\, U'(u)$$

$$E(u_-, u_+) = -\frac{f(u_-) - f(u_+)}{u_- - u_+}\Big(U(u_+) - U(u_-)\Big) + F(u_+) - F(u_-)$$
$$\leq 0$$

Zero entropy dissipation function $\varphi_0^\flat : \mathbb{R} \mapsto \mathbb{R}$.

$$E(u, \varphi_0^\flat(u)) = 0, \qquad \varphi_0^\flat(u) \neq u \quad (\text{ when } u \neq 0)$$
$$(\varphi_0^\flat \circ \varphi_0^\flat)(u) = u.$$



Solving the Riemann problem.

$$u(x,0) = \begin{cases} u_l, & x < 0 \\ u_r, & x > 0 \end{cases}$$

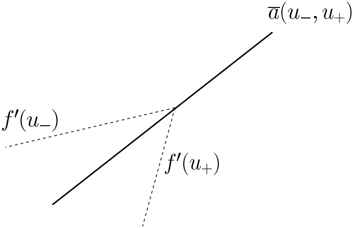
A single entropy inequality allows for:

- Classical compressive shocks

$$u_- > 0, \qquad \varphi^\natural(u_-) \le u_+ \le u_-$$

satisfying Lax shock inequalities

$$f'(u_-) \ge \frac{f(u_+) - f(u_-)}{u_+ - u_-} \ge f'(u_+)$$



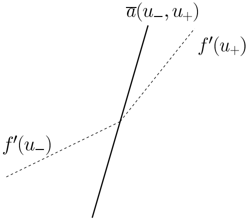
- Nonclassical undercompressive shocks

$$u_- > 0, \qquad \varphi_0^\flat(u_-) \le u_+ \le \varphi^\natural(u_-)$$

having all characteristics passing through

$$\min\Big(f'(u_-), f'(u_+)\Big) \ge \frac{f(u_+) - f(u_-)}{u_+ - u_-}$$

The cord connecting u_- to u_+ intersects the graph of f.



- Rarefaction waves. Lipschitz continuous solutions u connecting two constant states and depending only upon $\xi = x/t$

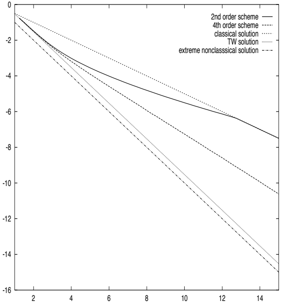
Entropy-compatible kinetic functions.

- A monotone decreasing, Lipschitz continuous function $\varphi^\flat : \mathbb{R} \mapsto \mathbb{R}$

$$\varphi_0^\flat(u) < \varphi^\flat(u) \leq \varphi^\natural(u), \quad u > 0$$

- Then, by definition, for each given left-hand state u_- the kinetic relation $u_+ = \varphi^\flat(u_-)$ singles out a nonclassical shock.

- Notation: Companion (threshold) function $\varphi^\sharp : \mathbb{R} \to \mathbb{R}$



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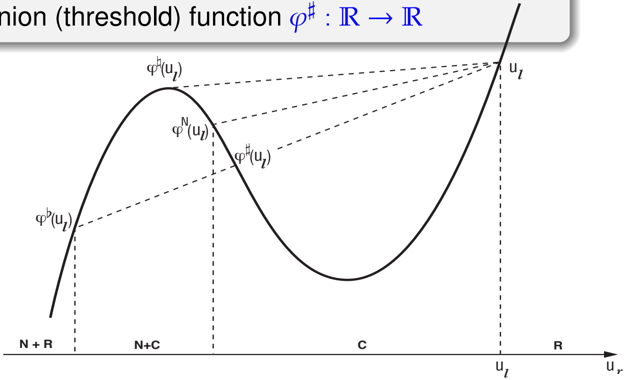
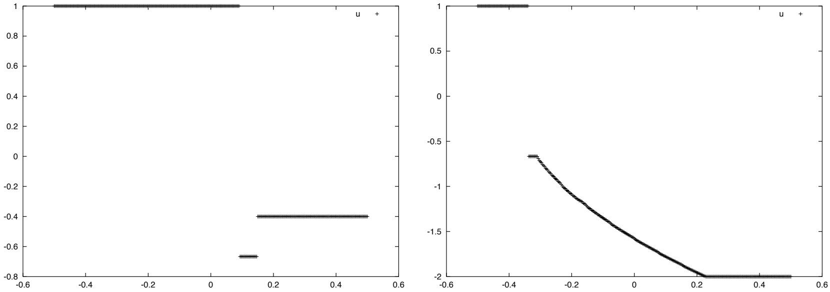


Nonclassical Riemann solver. For instance, suppose $u_l > 0$.

- $u_r \geq u_l$: rarefaction wave.
- $u_r \in [\varphi^\sharp(u_l), u_l)$: classical shock.
- $u_r \in (\varphi^\flat(u_l), \varphi^\sharp(u_l))$: nonclassical shock $\big(u_l, \varphi^\flat(u_l)\big)$ + classical shock $\big(\varphi^\flat(u_l), u_r\big)$.
- $u_r \leq \varphi^\flat(u_l)$: nonclassical shock $\big(u_l, \varphi^\flat(u_l)\big)$ + rarefaction wave.



THE NONCLASSICAL RIEMANN SOLVER BASED ON AN ENTROPY-SATISFYING KINETIC FUNCTION

Given a kinetic function $\varphi^\flat$ compatible with an entropy U of a conservation law with concave/convex flux, the Riemann problem admits a unique solution, satisfying:

- the single entropy inequality
- the kinetic relation $u_+ = \varphi^\flat(u_-)$ at each undercompressive shock

L^1 continuous dependence property ($t \in [0, T]$, compact $K \subset \mathbb{R}$)

$$\|u(t) - v(t)\|_{L^1(K)} \le C(T, K)\, \|u(0) - v(0)\|_{L^1(K)}$$

Generalizations

- 2×2 isentropic Euler equations and nonlinear elasticity or phase transition system

  uniqueness if hyperbolic, non-uniqueness if hyperbolic-elliptic)
  (Shearer et al., LeFloch, PLF-Thanh)

- $N \times N$ strictly hyperbolic systems of conservation laws.

  B.T. Hayes and P.G. LeFloch, Nonclassical shock waves and kinetic relations. Strictly hyperbolic systems, SIAM J. Math. Anal. (2000).

3. KINETIC FUNCTIONS BASED ON TRAVELING WAVES

For instance, consider conservation laws with nonlinear diffusion and linear dispersion

$$u_t + f(u)_x = \beta \left(|u_x|^p \, u_x \right)_x + u_{xxx}$$

$$f \text{ concave-convex}, \quad \beta > 0, \quad p \geq 0$$

Internal structure of shock waves:

- second-order ODE for traveling wave solutions $u(x,t) = u(y)$ with $y = x - \lambda t$

$$-\lambda (u - u_-) + f(u) - f(u_-) = \beta \, |u'|^p \, u' + u''$$

- with boundary conditions

$$\lim_{y \to \pm\infty} u(y) = u_\pm$$

- prescribed data $u_\pm, \lambda$ satisfying the Rankine-Hugoniot relation

Three regimes.

- $\beta \in (0, +\infty)$: diffusion and dispersion kept in balance
- $\beta = 0$: dispersion only
- $\beta \to +\infty$: diffusion only

First results.

For the cubic flux $f = u^3$, one has $\varphi^\natural(u) = -u/2$, $\varphi_0^\flat(u) = -u$, and closed formulas are available:

- $p = 0$: Shearer et al. (1995)
  $\varphi^\flat$ is piecewise linear (with slope -1 and $-1/2$)
- $p = 1/2$: Bedjaoui - PLF
  $\varphi^\flat$ is linear with slope $c_\beta \in (-1/2, -1)$
- $p = 1$: Hayes - PLF (1997)
  $\varphi^{\flat\prime}(0) = \varphi_0^\flat(0) = -1$

Phase plane analysis. (LeFloch-Bedjaoui, 2001 & 2004)

- existence of classical / nonclassical traveling waves
- Kinetic function $\varphi^\flat$ associated to this model ?
- Monotonicity ?
- Behavior near $u = 0$?

KINETIC FUNCTIONS BASED ON TRAVELING WAVES

To a large class of augmented models, we are able to associate a unique kinetic function which is monotone and satisfies the assumptions required in the theory of the Riemann problem.

Generalizations.

- 2×2 Nonlinear elasticity/Euler equations (non-nec. monotone) (Shearer et al., PLF-Bedjaoui)
- 2×2 Van de Waals model (two inflection points, multiple solutions) (Bedjaoui-Chalons-Coquel-PLF)

Admissible shocks

$$S(u_-) = \left\{ u_+ \ / \ \text{there exists a TW connecting } u_\pm \right\}$$

Theorem. (Bedjaoui - PLF, 2001 & 2004).
(i) Kinetic function $\varphi^\flat : \mathbb{R} \to \mathbb{R}$, Lipschitz continuous, strictly decreasing,

$$S(u) = \left\{ \varphi^\flat(u) \right\} \cup \left(\varphi^\sharp(u), u \right], \qquad u > 0$$

$$\varphi_0^\flat(u) < \varphi^\flat(u) \leq \varphi^\natural(u), \qquad u > 0$$

(ii) Threshold function $A^\natural$ such that

- $0 \leq p \leq 1/3$:

$$A^\natural : \mathbb{R} \to [0, \infty) \text{ Lipschitz continuous}, \quad A^\natural(0) = 0$$

$$\varphi^\flat(u) = \varphi^\natural(u) \quad \text{iff} \ \ \beta \geq A^\natural(u)$$

- $p > 1/3$:

$$\varphi^\flat(u) \neq \varphi^\natural(u) \qquad (u \neq 0)$$

(iii) Asymptotic behavior of infinitesimally small shocks:

- $p = 0$: $\varphi^{\flat\prime}(0) = \varphi^{\natural\prime}(0) = -1/2$
  $A^{\natural}(0) = 0, \qquad A^{\natural\prime}(0\pm) \neq 0$
- $0 < p \leq 1/3$: $\varphi^{\flat\prime}(0) = -1/2$
  $A^{\natural}(0) = 0, \quad A^{\natural\prime}(0\pm) = +\infty$
- $1/3 < p < 1/2$: $\varphi^{\flat\prime}(0) = -1/2$
- $p = 1/2$: $\varphi^{\flat\prime}(0) \in \left(\varphi_0^{-\flat\prime}(0), -1/2\right) = (-1, -1/2)$

$$\lim_{\beta \to 0+} \varphi^{\flat\prime}(0) = -1, \qquad \lim_{\beta \to +\infty} \varphi^{\flat\prime}(0) = -1/2$$

- $p > 1/2$: $\varphi^{\flat\prime}(0) = -1$

4. THE INITIAL VALUE PROBLEM

The behavior of the kinetic function for arbitrarily small shocks is required in our general existence theory.

Glimm method.

- Nonclassical Riemann solver as building block
- Random choice (equidistributed) or front tracking technique
- Numerical experiments (Chalons - PLF 2003)

EXISTENCE THEORY FOR THE INITIAL VALUE PROBLEM

- Theoretical convergence results in the strong L^1 norm
- Uniform convergence at points of continuity
- Convergence of left- and right-hand limit at discontinuities
- $TV(u(t,\cdot))$ is uniformly bounded (but need not be decreasing, generalized total variation functionals adapted to nonclassical wave interactions)

PLF, Hyperbolic Systems of Conservation Laws. The theory of classical and nonclassical shock waves, Birkhäuser (2002)

Baiti - PLF - Piccoli 2001, LeFloch 2002, PLF - Laforest 2010, 2015)

5. SCHEMES WITH WELL-CONTROLED DISSIPATION

- Consider the limiting solutions $u_\alpha = \lim_{\varepsilon \to 0} u_{\alpha\varepsilon}$ to a diffusive -dispersive conservation law

$$\partial_t u + \partial_x f(u) = \varepsilon\, u_{xx} + \alpha\, \varepsilon^2\, u_{xxx}, \qquad u = u_{\alpha,\varepsilon}$$

together with the associated kinetic function $\varphi^\flat_\alpha$

- Consider numerical solutions $u_\alpha^{\Delta x}$ given by some finite difference schemes together with its limit $v_\alpha = \lim_{\Delta x \to 0} u_\alpha^{\Delta x}$ and its associated kinetic function $\psi^\flat_\alpha$

Essential observation $v^\alpha \neq u^\alpha \qquad \psi^\flat_\alpha \neq \varphi^\flat_\alpha$

- schemes in conservative form, satisfying a discrete version of the entropy inequality (in the sense of Lax)
- small-scale effects drive the selection of shocks
- discrete dissipation $\neq$ continuous dissipation

Hayes-LeFloch criterion

$\psi^\flat_\alpha$ should be an accurate approximation of $\varphi^\flat_\alpha$

Hayes & PLF, Nonclassical shocks and kinetic relations. Finite difference schemes, SIAM Journal of Numerical Analysis (1998)

Schemes with well-controled dissipation.

- Finite difference schemes in conservative form in the sense of Lax
- Entropy conservative flux associated with the hyperbolic system
  - High-order accurate
  - Discrete version of the physically relevant entropy inequality
- High-order finite differences for the augmented terms (viscosity, capillarity, etc.), preserving the discrete entropy inequality
- Essential requirement: the equivalent equation (also called the modified equation) should coincide with the augmented physical model, with very high accuracy.

For instance for $p \geq 3$ (at least)

$$\partial_t u + \partial_x f(u) = \epsilon\, u_{xx} + \alpha\, \epsilon^2\, u_{xxx}$$

$$\partial_t u + \partial_x f(u) = \Delta x\, u_{xx} + \alpha\, (\Delta x)^2\, u_{xxx} + \mathcal{O}(\Delta x)^p$$

where we wrote $u_j^n = u(t_n, x_j) = u(n\Delta t, j\Delta x)$ and *formally* expanded in $\Delta t, \Delta x \to 0$

A conjecture about the equivalent equation.

- PLF : *As $p \to \infty$ the kinetic function $\psi^\flat_{\alpha,p}$ associated with a scheme with equivalent equation*

$$\partial_t u + \partial_x f(u) = \Delta x\, u_{xx} + \alpha\, (\Delta x)^2\, u_{xxx} + O(\Delta x)^p$$

converges to the exact kinetic function $\varphi^\flat_\alpha$

$$\lim_{p\to\infty} \psi^\flat_{\alpha,p} = \varphi^\flat_\alpha$$

References.

- Hayes - PLF (SINUM, 1998) scalar conservation laws
- PLF - Rohde (SINUM, 2000) third and fourth order schemes
- Chalons - PLF (JCP, 2001) van der Waals fluids
- PLF - Mohamadian (JCP, 2008) very high-order schemes
- Review paper: *PLF and Mishra, Numerical methods with controled dissipation for small-scale dependent shocks, Acta Numerica 23 (2014)*

Class of $2p$-th order WCD schemes

$$\frac{du_i}{dt} + \frac{1}{\Delta x}\sum_{j=-p}^{j=p} \alpha_j f_{i+j} = \frac{c}{\Delta x}\sum_{j=-p}^{j=p} \beta_j u_{i+j} + \alpha\frac{c^2}{\Delta x}\sum_{j=-p}^{j=p} \gamma_j u_{i+j}$$

$2p$-*order accuracy* for al $0 \le l \le 2p$

$$\sum_{j=-p}^{p} j\alpha_j = 1, \qquad \sum_{j=-p}^{p} j^l\alpha_j = 0, \qquad l \neq 1$$

$$\sum_{j=-p}^{p} j^2\beta_j = 2, \qquad \sum_{j=-p}^{p} j^l\beta_j = 0, \qquad l \neq 2$$

$$\sum_{j=-p}^{p} j^3\gamma_j = 6, \qquad \sum_{j=-p}^{p} j^l\gamma_j = 0, \qquad l \neq 3$$

Stability Condition on c (ensures good approximation for shocks of large strength)

$$\sigma < \alpha c^2 + c$$

where σ is the local wave speed.

$(2p+1)$-point, conservative, semi-discrete schemes

$$\frac{d}{dt}u_j = -\frac{1}{h}\left(g^*_{j+1/2} - g^*_{j-1/2}\right)$$

- $u_j = u_j(t)$ is an approximation of $u(x_j, t)$, and $h > 0$ is the mesh length
- The discrete flux

$$g^*_{j+1/2} = g^*(v_{j-p+1}, \cdots, v_{j+p}), \qquad v_j = \nabla U(u_j)$$

must be consistent with the exact flux g

$$g^*(v, \dots, v) = g(v).$$

Entropy conservative flux.

- Second-order entropy conservative flux — Tadmor 1984
- Third-order entropy conservative flux — LeFloch-Rohde 2000
- Arbitrarily high order, discrete in time — LeFloch-Mercier-Rohde 2000

Theorem (Second-order, Tadmor, 1984). Two-point numerical flux

$$g^*(v_0, v_1) = \int_0^1 g(v_0 + s\,(v_1 - v_0))\,ds, \qquad v_0, v_1 \in \mathbb{R}^N$$

where v is the entropy variable associated with a strictly convex entropy.

- Entropy conservative scheme, satisfying

$$\frac{d}{dt}U(u_j) + \frac{1}{h}\Big(G^*_{j+1/2} - G^*_{j-1/2}\Big) = 0$$

with

$$G^*(v_0, v_1) = \frac{1}{2}(G(v_0) + G(v_1)) + \frac{1}{2}(v_0 + v_1)\,g^*(v_0, v_1) - \frac{1}{2}\Big(v_0 \cdot g(v_0) + v_1 \cdot g(v_1)\Big)$$

- Second-order accurate, with (conservative) equivalent equation

$$\partial_t u + \partial_x f(u) = \frac{h^2}{6}\,\partial_x\Big(-g(v)_{xx} + \frac{1}{2}\,v_x \cdot \partial_x Dg(v)\Big)$$

Theorem. (Third-order, PLF - Rohde, 2000) Given any symmetric $N \times N$ matrices $B^*(v_{-p+2}, \cdots, v_p)$, the $(2p+1)$-point scheme associated with

$$g^*(v_{-p+1}, \cdots, v_p) = \int_0^1 g(v_0 + s\,(v_1 - v_0))\,ds - \frac{1}{12}\Big((v_2 - v_1)\cdot B^*(v_{-p+2}, \cdots, v_p) - (v_0 - v_{-1})\cdot B^*(v_{-p+1}, \cdots, v_{p-1})\Big)$$

is entropy conservative, with entropy flux

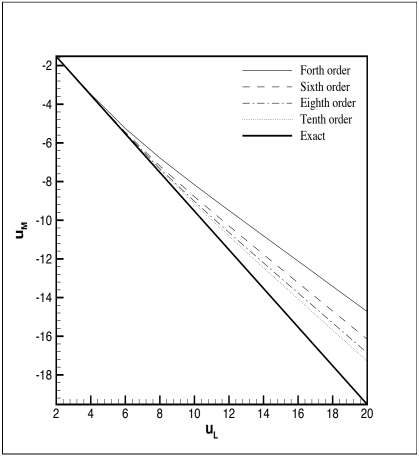
$$G^*(v_{-p+1}, \cdots, v_p) = \frac{1}{2}(v_0 + v_1)\cdot g^*(v_{-p+1}, \cdots, v_p) - \frac{1}{2}\Big(\psi^*(v_{-p+2}, \cdots, v_p) + \psi^*(v_{-p+1}, \cdots, v_{p-1})\Big).$$

When $p = 2$ and $B^*(v, v, v) = B(v)\Big(= Dg(v)\Big)$, this five-point scheme is third-order, at least.

6. COMPUTING KINETIC FUNCTIONS. 6.1 Cubic conservation law

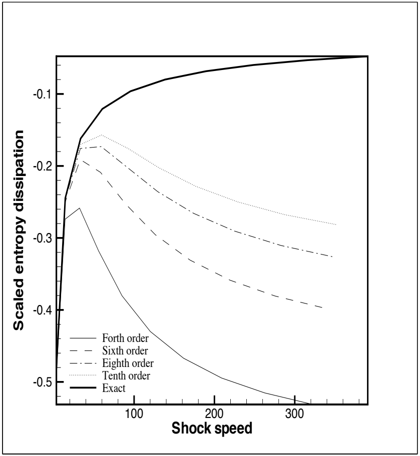
$$u_t + (u^3)_x = \varepsilon\, u_{xx} + \alpha\, \varepsilon^2\, u_{xxx}$$

Kinetic function



Scaled entropy dissipation

$\phi(s)/s^2$ (versus shock speed s)



6.2 The kinetic relation for the Camassa-Holm model

$$u_t + (u^3)_x = \varepsilon\, u_{xx} + \alpha\, \varepsilon^2 \big(u_{txx} + 2 u_x\, u_{xx} + u\, u_{xxx} \big)$$

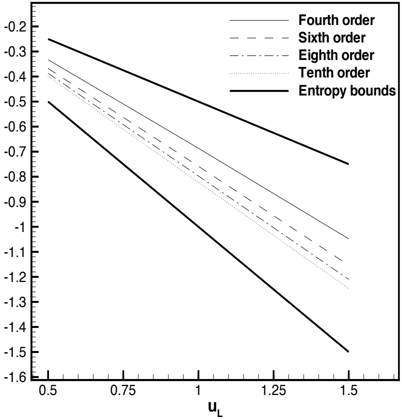
Theory.

- Well-posedness for the initial-value problem
  Bressan, Constantin, Karlsen, Coclite, Raynaud
- Kinetic relations via traveling wave analysis: open problem

Numerical investigation

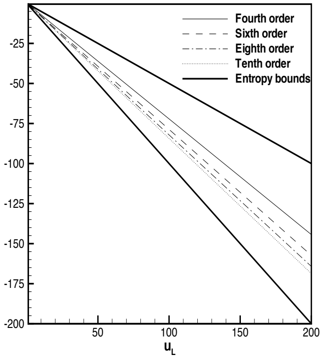
- Existence of a kinetic function ? Globally monotone ?
- Relation with the linear diffusive-dispersive model ?

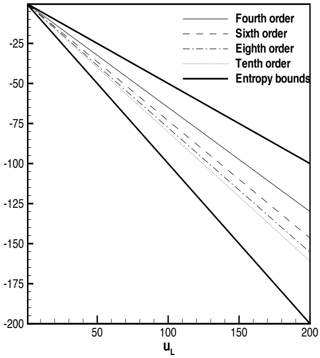
Shocks with moderate strength.



The kinetic functions for the linear diffusion-dispersion and Camassa-Holm models almost coincide for shocks with moderate strength.

Shocks with large strength.





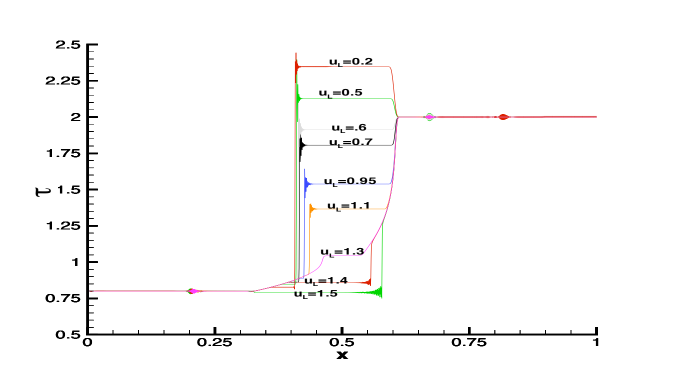
Linear diffusion-dispersion model

Camassa-Holm model

6.3 The kinetic relation for Van der Waals fluids

Complex wave structure.

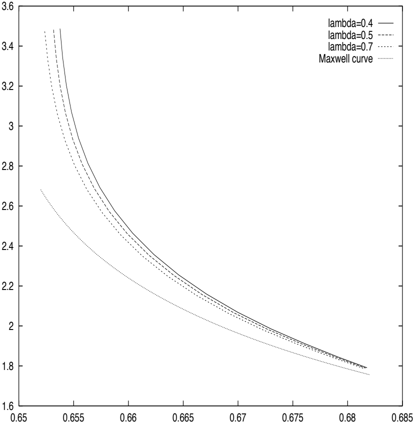
Initial data $\tau_L = 0.8$, $\tau_R = 2$, $u_R = 1$ with variable left-hand data u_L



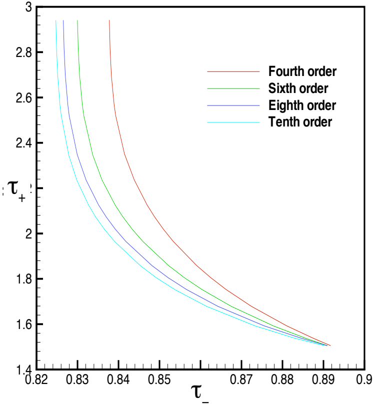
Better described... with the kinetic function

Kinetic function.

For τ near to 1: existence and monotonicity



Varying the capillarity coefficient



Varying the order of the discretization

Schemes with Well Controled Dissipation (WCD)

- Robust and reliable schemes, validated by an analysis of the equivalent equation
- Numerical kinetic function approaching (with arbitrary accuracy) the exact kinetic function $\lim_{p\to\infty} \psi^{\flat}_{\alpha,p} = \varphi^{\flat}_{\alpha}$
- Schemes based on entropy conservative flux, ensuring the correct sign on the entropy dissipation $U(u)_t + F(u)_x \leq 0$

Approximation of the nonclassical entropy solutions with arbitrary accuracy

- The kinetic function characterizes the shock dynamics and was investigated for a large class of models.
- Computing the kinetic function provides a tool.
  - Effect of the diffusion/dispersion ratio
  - Effect of the regularization
  - Order of accuracy of the scheme
  - Compare several physical models

7. THE ZERO-DIFFUSION-DISPERSION LIMIT

- **Navier-Stokes-Korteweg** system

$$\begin{aligned} \rho_t + (\rho u)_x &= 0 \\ (\rho u)_t + \left(\rho u^2 + p(\rho)\right)_x &= \varepsilon \left(\mu(\rho)\, u_x\right)_x + \varepsilon^2 \left(K[\rho]\right)_x \\ K[\rho] &= \rho\kappa(\rho)\rho_{xx} + \frac{1}{2}\left(\rho\kappa'(\rho) - \kappa(\rho)\right)\rho_x^2 \end{aligned}$$

- Convergence to the **Euler system** when $\varepsilon \to 0$

$$\begin{aligned} \rho_t + (\rho u)_x &= 0 \\ (\rho u)_t + \left(\rho u^2 + p(\rho)\right)_x &= 0 \end{aligned}$$

Rigorous convergence theorem for general functions $p(\rho), \mu(\rho), \kappa(\rho)$

- Asymptotic conditions: $p(\rho) \sim \rho^\gamma$, $\mu(\rho) \sim \rho^a$, $\kappa(\rho) \sim \rho^b$ when $\rho \to 0$ (vacuum) and $\rho \to +\infty$ (unbounded density)
- Coercivity condition on the capillarity $\kappa(\rho)$ (see below)

Germain-PLF, The finite energy method for compressible fluids. The Navier-Stokes-Korteweg model, Comm. Pure Appl. Math. (2015)

Notion of weak solutions

- **Finite total energy**

$$\sup_{t\geq 0} \int \Big(\frac{1}{2}\rho u^2 + \rho e(\rho) + \frac{1}{2}\kappa(\rho)\rho_x^2\Big)(t,x)\,dx < +\infty,$$

where the internal energy $e = e(\rho)$ is defined by $e'(\rho) := \frac{p(\rho)}{\rho^2}$.

- **Finite total effective energy**

$$\sup_{t\geq 0} \int \Big(\frac{1}{2}\rho \widetilde{u}^2 + \rho e(\rho) + \frac{1}{2}\kappa(\rho)\rho_x^2\Big)(t,x)\,dx < +\infty,$$

where the effective velocity is $\widetilde{u} = u + \frac{\mu(\rho)}{\rho^2}\rho_x$.

This class of weak solutions allows for shock waves, vacuum regions, unbounded fluid density, etc.

We have also some (singular) bounds on derivatives, deduced from the dissipation terms.

Effective Navier–Stokes–Korteweg system

- Given a constant ω, we define the ω-effective velocity and the ω-effective capillarity

$$\widetilde{u}^\omega = u + \omega \frac{\mu(\rho)}{\rho^2}\rho_x, \qquad \widetilde{\kappa}^\omega = \kappa - \omega(1-\omega)\frac{\mu^2}{\rho^3}$$

- If $\rho, u)$ is a solution to NSK, then $(\rho, \widetilde{u}^\omega)$ solves

$$\rho_t + (\rho\widetilde{u}^\omega)_x = \left(\omega\frac{\mu(\rho)}{\rho}\rho_x\right)_x$$
$$(\rho\,\widetilde{u}^\omega)_t + \left(\rho(\widetilde{u}^\omega)^2 + p(\rho)\right)_x = M^\omega[\rho,u]_x + K^\omega[\rho]_x$$

with modified viscosity and capillarity

$$M^\omega[\rho,u] = \frac{\mu(\rho)}{\rho}\left((1-\omega)\,\rho\,\widetilde{u}^\omega_x + \omega\,\rho_x\,\widetilde{u}^\omega\right)$$
$$K^\omega[\rho] = \rho\,\widetilde{\kappa}^\omega(\rho)\rho_{xx} + \frac{1}{2}\left(\rho\widetilde{\kappa}^{\omega\prime}(\rho) - \widetilde{\kappa}^\omega(\rho)\right)\rho_x^2$$

Case $\omega = 1$: Bresch-Desjardins, Mellet-Vasseur. Case $\omega = 1/2$: Jüngel

The strong coercivity condition

- **Effective energy balance law**

$$\widetilde{\mathscr{E}}[\rho,u](t)+\int_0^t \widetilde{\mathscr{D}}[\rho,u](s)\,ds=\widetilde{\mathscr{E}}[\rho,u](0)$$

$$\widetilde{\mathscr{D}}[\rho,u](t)=\int_{\mathbb{R}}\left(\frac{\mu(\rho)}{\rho^2}p'(\rho)\rho_x^2+\frac{\mu(\rho)}{\rho}\kappa(\rho)\Big(\rho_{xx}^2+\zeta(\rho)\rho_x^4\Big)\right)dx$$

$$-3\zeta=\frac{1}{2}\frac{\kappa''}{\kappa}+\left(\frac{\mu}{\rho}\right)''\frac{\rho}{\mu}$$

- **Strong coercivity** (SC) condition

There exists $C_0>0$ such that for all function $\rho=\rho(x)>0$

$$(SC)\qquad \widetilde{\mathscr{D}}[\rho,u](t)\geq C_0\int\left(\rho_{xx}^2+\frac{\rho_x^4}{\rho^2}\right)\frac{\mu(\rho)\kappa(\rho)}{\rho}\,dx$$

The family of entropy pairs

- The **key compactness property**

  NOT based on a "Sobolev embedding theorem" (a bound on high-order derivatives implies a compactness property for low-order derivatives), but:

> based on the STRUCTURE of the Euler equations and an INFINITE list of BALANCE LAWS. (DiPerna 1985, Germain-PLF 2015)

- **Balance laws** generated from a Green kernel (χ, σ)

$$\left(\eta^\psi(\rho, u)\right)_t + \left(q^\psi(\rho, u)\right)_x \leq \mathcal{O}(\varepsilon) \qquad \text{for all convex } \psi = \psi(v)$$

$$(\eta^\psi, q^\psi)(\rho, u) = \int_{\mathbb{R}} (\chi, \sigma)(\rho, u, v)\, \psi(v)\, dv$$

  - **Polytropic** $p(\rho) = k\rho^\gamma$

$$\chi(\rho, u, v) = \left(\rho^{\gamma-1} - (v-u)^2\right)_+^{\frac{3-\gamma}{2(\gamma-1)}}$$

$$\sigma(\rho, u, v) = \left(u + \frac{\gamma-1}{2}(v-u)\right)\left(\rho^{\gamma-1} - (v-u)^2\right)_+^{\frac{3-\gamma}{2(\gamma-1)}}$$

  - **Real fluids** $p(\rho) \simeq k\rho^\gamma$: nonlinear superposition formula (expansion in a series of Bessel functions)