Analytical evaluation of 3D BEM integral representations using complex analysis

Sonia Mogilevskaya

Department of Civil, Environmental, and Geo-Engineering
University of Minnesota

January 15, 2015
Boundary Element Methods

Engineering Problems as Boundary Value Problems

The Boundary Element Methods represent a family of general numerical techniques for solving a large class of engineering problems that can be reduced to the mathematical boundary value problems.

**Boundary Value Problem**

- Prescribed data at the boundary
- Region governed by a differential equation
- Initial conditions for time-dependent problems
Boundary Element Methods

Main Idea

Governing differential equation

Integral representations

describe the fields at the point $M$ via the integrals over some data at the boundary $S$

Boundary Integral equations

$M \rightarrow M_0 \in S$

describe the fields at the point $M_0$ on the boundary via the integrals over prescribed boundary data
BEM Structure

Boundary Value Problem

Fundamental solution

Indirect BEM
- Potentials (integrals of fundamental solutions)

Direct BEM
- Integral identities (e.g., reciprocal theorem)

Boundary Integral Equation

exactly satisfies governing equation everywhere but one point

express the fields in the domain via boundary data and fundamental solutions
Integral Representations: Laplace Equation

\[ \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \]

Fundamental solution

\[ G(x, \xi) = \frac{1}{4\pi r} \]

\[ \nabla^2 G(x, \xi) = 0, \quad r \neq 0 \]

Integral Representations

Single-layer potential

\[ u(x) = \frac{1}{4\pi} \int_S \frac{\varphi(\xi)}{r} dS_\xi \]

\[ \frac{1}{4\pi} \frac{\partial}{\partial n(x)} \int_S \varphi(\xi) \frac{1}{r} dS_\xi \]

Double-layer potential

\[ u(x) = \frac{1}{4\pi} \int_S \psi(\xi) \frac{\partial}{\partial n(\xi)} \frac{1}{r} dS_\xi \]

\[ \frac{1}{4\pi} \frac{\partial}{\partial n(x)} \int_S \psi(\xi) \frac{\partial}{\partial n(\xi)} \frac{1}{r} dS_\xi \]
Integral Representations: Helmholtz Equation

\[ \nabla^2 u + k^2 u = 0, \quad u = u(x, \omega) = \text{Re}[u(x) \exp(-i\omega r)], \quad k = \omega / c \]

- Wave number
- Frequency
- Sound speed

Fundamental solution

\[ G(x, \xi) = \frac{1}{4\pi} \frac{1}{r} \exp(ikr) \]

Integral Representations

**Single-layer potential**

\[ u(x) = \frac{1}{4\pi} \int_S \frac{\varphi(\xi)}{r} \exp(ikr) dS_{\xi} \]

\[ \frac{1}{4\pi} \frac{\partial}{\partial n(x)} \int_S \varphi(\xi) \frac{1}{r} \exp(ikr) dS_{\xi} \]

**Double-layer potential**

\[ u(x) = \frac{1}{4\pi} \int_S \frac{\psi(\xi)}{r} \frac{\partial}{\partial n(\xi)} \left[ \frac{1}{r} \exp(ikr) \right] dS_{\xi} \]

\[ \frac{1}{4\pi} \frac{\partial}{\partial n(x)} \int_S \frac{\psi(\xi)}{r} \frac{\partial}{\partial n(\xi)} \left[ \frac{1}{r} \exp(ikr) \right] dS_{\xi} \]
Integral Representations: Navier-Cauchy Equation

\[ \lambda u_{k,km} + \mu \left( u_{m,kk} + u_{k,km} \right) = 0 \]

\[ u_{,k} = \partial u / \partial x_k \]

**Fundamental solution**

\[ U_{mj}(x,\xi) = \frac{1}{16\pi\mu (1-\nu)} \left[ (3-4\nu) \delta_{mj} + r_{,m} r_{,j} \right] \]

**Integral Representations**

**Single-layer potential**

\[ u(x) = \frac{1}{4\pi} \int_S \varphi(\xi) U(x,\xi) dS_\xi \]

\[ t(x) = \frac{1}{4\pi} \int_S \varphi(\xi) T(x,\xi) dS_\xi \]

**Double-layer potential**

\[ u(x) = \frac{1}{4\pi} \int_S \psi(\xi) T^T(\xi,x) dS_\xi \]

\[ t(x) = \frac{1}{4\pi} \int_S \psi(\xi) H(x,\xi) dS_\xi \]

Region of interest \( S \)
Numerical Solution

Discretization of the boundary (elements)

Approximation of the functions

Evaluation of the integrals

Final system of equations

Solution of the system of equations

Computation of the fields at the boundary and inside the domain
Evaluation of Integrals

Isoparametric elements

\[ x^{(e)} \approx \sum_q \Phi_q^{(e)} x_q^{(e)} \]

\[ u^{(e)} \approx \sum_m \Phi_m^{(e)} u_m^{(e)} \]

Integrals

\[ \int_{S^{(e)}} \Phi_m G(x, \xi) dS_x, \int_{S^{(e)}} \Phi_m \frac{\partial G(x, \xi)}{\partial n_x} dS_x \]

Fundamental solutions (scalar or vector functions)

Most consuming part of the BEM procedure!
Why Analytical Integration?

- Singular, hypersingular, and near singular integrals appear in the “limit before integration procedure.”
- Lack of reliable quadrature rules.
- Analytical integration leads to higher accuracy of computation and to the reduction of its cost.
- Analytical integration may facilitate the use of the so-called fast methods (e.g. fast multipole method) for solving large systems of algebraic equations.
- Analytical integration routines can be used as “black boxes” by the developers of the BEM-based software.
Wirtinger Calculus

New independent variables

\[ z = x + iy, \quad \bar{z} = x - iy \]

\[ f(x, y) \quad \rightarrow \quad f(z, \bar{z}) \]

Interrelations between variables

\[ x = \frac{z + \bar{z}}{2}, \quad y = \frac{z - \bar{z}}{2i} \]

Wirtinger derivatives

\[ \frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right); \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \]
Complex Integral Theorems

Green’s representation formula

\[ \frac{1}{4} \int_{\Sigma} \left( f\nabla^2 g - g\nabla^2 f \right) \, dz \, d\bar{z} = -\int_{\partial \Sigma} \left( f \frac{\partial g}{\partial z} \, dz + g \frac{\partial g}{\partial \bar{z}} \, d\bar{z} \right) \]

Gauss theorems

\[ \int_{\Sigma} \frac{\partial f}{\partial \bar{z}} \, dS = -\frac{i}{2} \int_{\partial \Sigma} f \, dz, \quad \int_{\Sigma} \frac{\partial f}{\partial z} \, dS = \frac{i}{2} \int_{\partial \Sigma} f \, d\bar{z} \]

Cauchy-Pompeiu formulae

\[ \frac{1}{2\pi i} \int_{\partial \Sigma} \frac{f(\tau)}{\tau - z} \, d\tau - \frac{1}{\pi} \int_{\Sigma} \frac{\partial f(\tau)}{\partial \tau} \frac{dS}{\tau - z} = \begin{cases} f(z) & z \in \Sigma \\ 0 & z \notin \Sigma \cup \partial \Sigma \end{cases} \]

\[ -\frac{1}{2\pi i} \int_{\partial \Sigma} \frac{f(\tau)}{\tau - z} \, d\bar{\tau} - \frac{1}{\pi} \int_{\Sigma} \frac{\partial f(\tau)}{\partial \tau} \frac{dS}{\tau - z} = \begin{cases} f(z) & z \in \Sigma \\ 0 & z \notin \Sigma \cup \partial \Sigma \end{cases} \]

Dimitrie Pompeiu

1873 – 1954
Complex Notations for Plane Elements

Complex notation geometry

\[ r^2 = (\tau - z)(\bar{\tau} - \bar{z}) + h^2, \quad dS_{\xi} = \left(\frac{i}{2}\right) d\tau d\bar{\tau} \]

\[ n = n_1(x) + i n_2(x), \quad n_3(x) \]

\[ \tau = \xi_1 + i \xi_2, \quad z = x_1 + i x_2, \quad h = \xi_3 - x_3 \]

Complex notations for the fields (elasticity)

\[ t = t_1(x) + i t_2(x), \quad t_3(x) \]

\[ u = u_1(\xi) + i u_2(\xi), \quad u_3(x) \]
Generic Integrals

Potential and Elasticity Problems

\[ \int_{s} \frac{(\tau - z)^m (\bar{\tau} - \bar{z})^n}{r} dS_{\xi}, \quad m, n = 0, 1, \ldots \]

Acoustics

\[ \int_{s} \frac{(\tau - z)^m (\bar{\tau} - \bar{z})^n}{r} \exp(ikr) dS_{\xi}, \quad m, n = 0, 1, \ldots \]

+ their derivatives of various orders over \( z, \bar{z}, h, k \)
Potential and Elasticity Problems

\[
\int \left( \frac{r}{S} \right) S \int dS \xi \frac{\left( \tau - z \right)^m \left( \tau - z \right)^n}{\tau - z}, \ m, n = 0, 1, \ldots
\]

can be re-written as

\[
\int S \frac{\partial f(z)}{\partial z} \frac{dS}{\tau - z}
\]

where

\[
f(z) = \begin{cases} 
-2\left( r / h^2 \right) \left( \tau - z \right)^{m+1} \left( \tau - z \right)^{n+1} \ _2F_1 \left( 1, n + 3 / 2, 3 / 2, r^2 / h^2 \right) & h \neq 0 \\
-2r(2n + 1)^{-1} \left( \tau - z \right)^m \left( \tau - z \right)^n & h = 0
\end{cases}
\]

hypergeometric functions
Acoustics

\[ \int_{S} \frac{\left(\tau - z\right)^{m} \left(\bar{\tau} - \bar{z}\right)^{n}}{r} \exp(ikr) \mathrm{d}S_{\xi}, \ m, n = 0, 1, \ldots \]

can be re-written as

\[ \int_{S} \frac{\partial f(z)}{\partial z} \frac{\mathrm{d}S_{\xi}}{\tau - z} \]

where

\[ \frac{\partial f(z)}{\partial z} = \frac{\left(\tau - z\right)^{m+1} \left(\bar{\tau} - \bar{z}\right)^{n}}{r} \exp(ikr) \]
Reduction of Generic Integral to a Contour Integral

Potential and Elasticity Problems

\[
\int_S \frac{(\tau - z)^m (\bar{\tau} - \bar{z})^n}{r} \, dS_{\xi} = \frac{1}{2i} \int f(\tau) d\tau - f(z) \left\{ \begin{array}{ll}
\pi & z \in S \\
\gamma / 2 & z \in \partial S \\
0 & z \notin S
\end{array} \right.
\]

Constant, linear, and quadratic approximation

\[
f(\tau) = \begin{cases} 
2r(\tau - z)^m & n = 0 \\
\frac{2}{3} r \left(r^2 - 3h^2\right)(\tau - z)^{m-1} & n = 1 \\
\frac{2}{15} r \left(3r^4 - 10h^2 r^2 + 15h^4\right)(\tau - z)^{m-2} & n = 2
\end{cases}
\]

\[r^2 = (\tau - z)(\bar{\tau} - \bar{z}) + h^2\]
Reduction of Generic Integral
to a Contour Integral

Acoustics

\[
\int_s \left( \frac{(\tau - z)^m (\bar{\tau} - \bar{z})^n}{r} \right) \exp(i kr) \, dS_\xi = \frac{1}{2i} \int_{\partial S} \frac{f(\tau) \, d\tau}{\tau - z} \left\{ \begin{array}{ll}
\pi & z \in S \\
\gamma / 2 & z \in \partial S \\
0 & z \notin S
\end{array} \right.
\]

Constant, linear, and quadratic approximation

\[
f(\tau) = \left\{ \begin{array}{ll}
2ik^{-1}(\tau - z)^m \exp(i kr) & n = 0 \\
2ik^{-3}(\tau - z)^{m-1} \exp(i kr) \left[ -2 + 2ikr + k^2 \left( r^2 - h^2 \right) \right] & n = 1 \\
2ik^{-5}(\tau - z)^{m-2} \exp(i kr) \left[ 24 + k \left( -8kh^2 - 24ir + k \left( r^2 - h^2 \right) \right) \left( k^2 \left( r^2 - h^2 \right) + 4ikr - 12 \right) \right] & n = 2
\end{array} \right.
\]

\[
r^2 = (\tau - z)(\bar{\tau} - \bar{z}) + h^2
\]
Analytical Evaluation of Generic Integral Potential and Elasticity Problems

Straight boundary segment \((a_j, a_{j+1})\)

Circular arc of radius \(R\) with the center \(Z_c\)

\[
\bar{\tau} = a_j + \frac{a_{j+1} - a_j}{a_{j+1} - a_j} (\tau - a_j)
\]

\[
\bar{\tau} = Z_c + \frac{R^2}{\tau - Z_c}
\]

Elementary integrals (evaluated in closed-form)

\[
\int_{\delta S} f\left(\tau, \bar{\tau}\right) \frac{d\tau}{\tau - Z}, f = \begin{cases} 
2r(\tau - z)^m & n = 0 \\
\frac{2}{3} r \left( r^2 - 3h^2 \right) (\tau - z)^{m-1} & n = 1 \\
\frac{2}{15} r \left(3r^4 - 10h^2 r^2 + 15h^4\right)(\tau - z)^{m-2} & n = 2
\end{cases}
\]

\[
\int_{a_j}^{a_{j+1}} r(\tau - z)^k d\tau, \quad -1 \leq k \leq 4
\]

\[
\int_{a_j}^{a_{j+1}} r\left(\tau - Z_c\right)^k d\tau, \quad -1 \leq k \leq 4
\]
Analytical Evaluation of Generic Integral

Acoustics

Straight boundary segment \((a_j, a_{j+1})\)

\[
\bar{\tau} = a_j + \frac{a_{j+1} - a_j}{a_{j+1} - a_j} (\tau - a_j)
\]

Circular arc of radius \(R\) with the center \(z_c\)

\[
\bar{\tau} = z_c + \frac{R^2}{\tau - z_c}
\]

Elementary integrals

\[
\int_{a_j}^{a_{j+1}} r^p (\tau - z)^k \exp(ikr) \, d\tau
\]

\[
\int_{a_j}^{a_{j+1}} r^p (\tau - z_c)^k \exp(ikr) \, d\tau, \quad -1 \leq k \leq 4
\]

In general cannot be evaluated in closed forms
Analytical Evaluation of Generic Integral Acoustics

Basic requirement for the BEM discretization: 5-10 elements per wavelength

Asymptotic expansion

$$\exp(ikr) = \exp(ikr_0) \left[ 1 + ik\delta r + \ldots + \frac{i^n}{n!} (k\delta r)^n + \ldots \right]$$

Resulting integrals

$$\int_{a_j}^{a_{j+1}} r^p \left( \tau - z_c \right)^k d\tau$$

Rapidly converging series
Example Elements

(a) 

(b) 

Diagram showing regions in the complex plane for different values of $\tau$.
Results: Potential & Elasticity Problems

\[ \int_{S} \frac{1}{r} dS, \quad h = 1 \]

Table 2  Comparison of numerical and analytical integration for \( \int_{S} dS/r, \quad h = 1 \) and a triangular element. The numerical results use \( N \times N \) Gauss points for three values of \( N \)

<table>
<thead>
<tr>
<th>( z )</th>
<th>Analytical</th>
<th>( N = 3 )</th>
<th>( N = 5 )</th>
<th>( N = 9 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2 + 0.4i</td>
<td>0.393223700304</td>
<td>0.393241498413</td>
<td>0.393223721363</td>
<td>0.393223700304</td>
</tr>
<tr>
<td>0.4 + 0.2i</td>
<td>0.471796231443</td>
<td>0.471865518258</td>
<td>0.471796605759</td>
<td>0.471796231458</td>
</tr>
<tr>
<td>0.66 + 0.33i</td>
<td>0.456198640156</td>
<td>0.456352741167</td>
<td>0.456199333137</td>
<td>0.456198640151</td>
</tr>
<tr>
<td>0.8 + 0.4i</td>
<td>0.436975043605</td>
<td>0.437112786145</td>
<td>0.436975423734</td>
<td>0.436975043588</td>
</tr>
<tr>
<td>4 + 2i</td>
<td>0.120426555085</td>
<td>0.120426563413</td>
<td>0.120426555084</td>
<td>0.120426555085</td>
</tr>
</tbody>
</table>
Results: Potential & Elasticity Problems

\[ \int_{S} \frac{1}{r} dS_{\xi}, h = 1 \]

Table 3  Comparison of numerical and analytical integration for \( \int_{S} dS/r, h = 1 \) and a circular-sector element. The numerical results use \( N \times N \) Gauss points for three values of \( N \)

<table>
<thead>
<tr>
<th>( z )</th>
<th>Analytical</th>
<th>( N = 3 )</th>
<th>( N = 9 )</th>
<th>( N = 20 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-0.2 + 0.4i)</td>
<td>0.64624210</td>
<td>0.64842122</td>
<td>0.64634795</td>
<td>0.64625259</td>
</tr>
<tr>
<td>(-0.4 - 0.2i)</td>
<td>0.53858071</td>
<td>0.54060724</td>
<td>0.53867925</td>
<td>0.53859048</td>
</tr>
<tr>
<td>(0.2 - 0.4i)</td>
<td>0.58590952</td>
<td>0.58854325</td>
<td>0.58603679</td>
<td>0.58592213</td>
</tr>
<tr>
<td>(0.4 + 0.2i)</td>
<td>0.72395773</td>
<td>0.72701346</td>
<td>0.72410217</td>
<td>0.72397203</td>
</tr>
<tr>
<td>(0.8 + 0.4i)</td>
<td>0.70127134</td>
<td>0.70456037</td>
<td>0.70142764</td>
<td>0.70128680</td>
</tr>
<tr>
<td>(4 + 2i)</td>
<td>0.19479573</td>
<td>0.19577969</td>
<td>0.19484158</td>
<td>0.19480026</td>
</tr>
</tbody>
</table>
Results: Acoustics

\[
\int_{S} \frac{1}{r} \exp(\mathit{ikr}) dS \xi, \ h = 0
\]

\[
\text{error\%} = 100 \times \frac{|\text{Num.}| - |\text{Anal.}|}{|\text{Num.}|}
\]

Num: Gauss quadrature 50x50 points

Table 1: Comparison of numerical and analytical integration (triangular element) for \( \eta = 5 \)

<table>
<thead>
<tr>
<th>( \int_{S} \frac{e^{ikr}}{r} dS )</th>
<th>( x_1 = (0, 0, 0) )</th>
<th>( x_2 = (0.25, 0.25, 0) )</th>
<th>( x_3 = (0.5, 0.5, 0) )</th>
<th>( x_4 = (0.75, 0.75, 0) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Numerical</td>
<td>1.1444 + 0.4214i</td>
<td>2.3062 + 0.4330i</td>
<td>1.6866 + 0.4306i</td>
<td>0.6683 + 0.4145i</td>
</tr>
<tr>
<td>Analytical</td>
<td>1.1444 + 0.4214i</td>
<td>2.3103 + 0.4330i</td>
<td>1.6896 + 0.4306i</td>
<td>0.6683 + 0.4145i</td>
</tr>
<tr>
<td>error%</td>
<td>0.01%</td>
<td>0.17%</td>
<td>0.17%</td>
<td>0.01%</td>
</tr>
</tbody>
</table>
Results: Acoustics

\[
\int_S \frac{1}{r} \exp(ikr) dS_y, \quad h = 0
\]

\[
\text{error}\% = 100 \times \frac{|\text{Num.}| - |\text{Anal.}|}{|\text{Num.}|}
\]

Num: Gauss quadrature 50x50 points

<table>
<thead>
<tr>
<th>[ \frac{e^{ikr}}{r} dS ]</th>
<th>(x_1 = (0, 0, 0))</th>
<th>(x_2 = (0.35, 0.35, 0))</th>
<th>(x_3 = (0.7071, 0.7071, 0))</th>
<th>(x_4 = (1.0, 1.0, 0))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Numerical</td>
<td>1.3758 + 0.6477i</td>
<td>2.9538 + 0.6778i</td>
<td>1.9662 + 0.6649i</td>
<td>0.7107 + 0.6222i</td>
</tr>
<tr>
<td>Analytical</td>
<td>1.3758 + 0.6477i</td>
<td>2.9561 + 0.6778i</td>
<td>1.9662 + 0.6649i</td>
<td>0.7107 + 0.6222i</td>
</tr>
<tr>
<td>error%</td>
<td>0.01%</td>
<td>0.0762%</td>
<td>0.01%</td>
<td>0.01%</td>
</tr>
</tbody>
</table>
Conclusions

• The approach provide closed form expressions for three-dimensional integrals involved in the integral representations of potential, elasticity, and acoustic scattering problems in case of planar elements bounded by straight segment and circular arcs

• All the integrals involved can be reduced to a few generic integrals and their derivatives with respect to specific parameters

• The proposed approach could be extended on the case of non-planar elements that could be mapped to planar ones using rational mapping functions

• The proposed approach can be extended to elastodynamic problems

• The approach could be employed to create integration subroutines or functions that can be used as “black boxes” by the developers of the BEM software

• The approach could potentially have applications in some FEM integral representations
Acknowledgements

This is a collaborative work with Dmitry Nikolskiy and Fatemeh Pourahmadian

Department of Civil, Environmental, and Geo Engineering, UMN

References

Potential and elasticity problems: with Dmitry Nikolskiy
“The use of complex integral representations for analytical evaluation of three-dimensional BEM integrals

Acoustics: with Fatemeh Pourahmadian
“Complex variables-based approach for analytical evaluation of three-dimensional integral representations