Nichols algebras of finite Gelfand-Kirillov dimension

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Joint work with N. Andruskiewitsch and I. Heckenberger

Plan of the talk

1. Gelfand-Kirillov dimension and pointed Hopf algebras.

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2. Nichols algebras of diagonal type.
3. Nichols algebras of blocks.
4. Nichols algebras of decomposable modules.
5. Pre-Nichols algebras and liftings.

Definition

Let A be a finitely generated $\mathbb{C}$-algebra. If V is a finite-dimensional generating subspace of A and $A_{V,n} = \sum_{0 \le j \le n} V^n$, then

$$\operatorname{GKdim} A := \overline{\lim}_{n\to\infty} \log_n \dim A_{V,n};$$

it does not depend on the choice of V. If A is not fin. gen., then

$$\operatorname{GKdim} A := \sup\{\operatorname{GKdim} B | B \subseteq A, \ B \text{ finitely generated}\}.$$

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Example

A commutative $\Rightarrow \operatorname{GKdim} A \in \mathbb{N}_0 \cup \infty$; if A fin. gen.,

$$\operatorname{GKdim} A = \operatorname{Krull\,dim} A = \dim \operatorname{Spec} A.$$

(Krull dim = sup of the lengths of all chains of prime ideals).

Problem

Classify Hopf algebras H with GKdim $H < \infty$.

Let $G = G(H) = \{x \in H - 0 : \Delta(x) = x \otimes x\}$.

H is *pointed* if $H_0 := \sum_{C \text{ simple subcoalgebra}} C = \mathbb{C}G(H)$.

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Example

Group algebras, enveloping algebras are pointed.

Remark

Let G be a finitely generated group. G is *virtually nilpotent* or *nilpotent-by-finite* if it has a normal nilpotent subgroup N such that G/N is finite. Then

$$\operatorname{GKdim} \mathbb{C}G < \infty \iff \text{growth } G < \infty \overset{\blacklozenge}{\iff} G \text{ virtually nilpotent.}$$

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Remark

Let A be an algebra with an ascending filtration. Then $\operatorname{GKdim} A \geq \operatorname{GKdim} \operatorname{gr} A$; $=$ holds if $\operatorname{gr} A$ is fg.

Thus, $\mathfrak{g}$ Lie algebra $\implies \operatorname{GKdim} U(\mathfrak{g}) = \dim \mathfrak{g}$.

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- R is connected and coradically graded.

${}^{\mathbb{C}G}_{\mathbb{C}G}\mathcal{YD}$ = category of Yetter-Drinfeld modules over $\mathbb{C}G$;

$V = \oplus_{g\in G} V_g$ is a G-graded vector space;
V is a left G-module such that $g \cdot V_h = V_{ghg^{-1}}$(compatibility).

Now $c \in GL(V \otimes V)$, $c(v \otimes w) = g \cdot w \otimes v$, for $v \in V_g$, $w \in V$, satisfies

$$(c \otimes \mathrm{id})(\mathrm{id} \otimes c)(c \otimes \mathrm{id}) = (\mathrm{id} \otimes c)(c \otimes \mathrm{id})(\mathrm{id} \otimes c);$$

so (V, c) is a braided vector space and ${}^{\mathbb{C}G}_{\mathbb{C}G}\mathcal{YD}$ is a braided tensor category.
$\rightsquigarrow$ we may consider Hopf algebras in ${}^{\mathbb{C}G}_{\mathbb{C}G}\mathcal{YD}$.

Definition

Given $V \in {}^{\mathbb{C}G}_{\mathbb{C}G}\mathcal{YD}$, the **Nichols algebra** of V is the graded Hopf algebra $\mathcal{B}(V) = \oplus_{n\geq 0}\mathcal{B}^n(V)$ in ${}^{\mathbb{C}G}_{\mathbb{C}G}\mathcal{YD}$ such that

$$\mathcal{B}^0(V) = \mathbb{C}, \qquad \mathcal{B}^1(V) = P(V), \qquad \mathcal{B}(V) = \mathbb{C}\langle \mathcal{B}^1(V)\rangle.$$

Hence $\mathcal{B}(V) \simeq T(V)/\mathcal{J}(V)$.

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$\mathcal{J}(V)$ maximal graded Hopf ideal generated by elements in $\oplus_{n\geq 2}\mathcal{B}^n(V)$.

Problem (**very difficult**)

Determine the ideal $\mathcal{J}(V)$ (it depends crucially on the braiding c on V).

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- $R = \oplus_{n\geq 0} R^n \in {}^{\mathbb{C}G}_{\mathbb{C}G}\mathcal{YD}$ coradically gr. connected Hopf alg. Then

$$\operatorname{GKdim} H < \infty \overset{(*)}{\Longrightarrow} \operatorname{GKdim} \operatorname{gr} H < \infty \overset{(**)}{\Longleftrightarrow} \begin{array}{r} \operatorname{GKdim} R < \infty, \\ \operatorname{GKdim} \mathbb{C}G < \infty. \end{array}$$

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- $V = R^1$: $\mathcal{B}(V) \hookrightarrow R$, so $\operatorname{GKdim} H < \infty \implies \operatorname{GKdim} \mathcal{B}(V) < \infty$.

Lifting Method (Andruskiewitsch-Schneider)

Let us fix a virtually nilpotent group G.

Question

Determine all $V \in {}^{\mathbb{C}G}_{\mathbb{C}G}\mathcal{YD}$ such that $\operatorname{GKdim} \mathcal{B}(V) < \infty$.

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Question

Compute all liftings of R, i.e. all H such that $\operatorname{gr} H \simeq R\#G$.

Γ **abelian group**, $V \in {}^{\mathbb{C}\Gamma}_{\mathbb{C}\Gamma}\mathcal{YD} \rightsquigarrow V = \oplus_{g\in\Gamma} V_g$, each $V_g \in \Gamma-$Mod.

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- **Point** of label $q \in \mathbb{C}^{\times}$: braided v. sp. (V, c) of dim. 1, $c = q\,\mathrm{id}$. $\mathbb{C}_g^{\chi} \in {}^{\mathbb{C}\Gamma}_{\mathbb{C}\Gamma}\mathcal{YD}$ of dimension 1, homogeneous of degree g, Γ acts by $\chi \in \widehat{\Gamma}$ and $\chi(g)$.

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- $\Gamma = \langle \mathbf{g} \rangle \simeq \mathbb{Z}$. $\mathcal{V}(\epsilon, \ell) \in {}^{\mathbb{C}\mathbb{Z}}_{\mathbb{C}\mathbb{Z}}\mathcal{YD}$ homogeneous of degree $\mathbf{g}$, dimension $\ell > 1$, the action of $\mathbf{g}$ given by a Jordan block of size ℓ and eigenvalue ϵ. These braided vector spaces are called **blocks**.

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- Every finite-dimensional indecomposable, not simple, in ${}^{\mathbb{C}\Gamma}_{\mathbb{C}\Gamma}\mathcal{YD}$ is a block as braided vector space.

(V, c) is of **diagonal type**: $\exists$ a basis $(x_i)_{i\in\mathbb{I}_\ell}$ and a matrix $\mathfrak{q} = (q_{ij})_{i,j\in\mathbb{I}_\theta}$, $q_{ii} \neq 1$ for all i, with connected diagram, such that

$$c(x_i \otimes x_j) = q_{ij}\, x_j \otimes x_i, \qquad\qquad i, j \in \mathbb{I}_\ell.$$

Question

Classify all braided vector spaces (V, c) of diagonal type with such that $\operatorname{GKdim} \mathcal{B}(V) < \infty$.

$$\mathfrak{q} \text{ is } \begin{cases} \text{generic:} & q_{ii}, q_{ij}q_{ji} \in \mathbb{C} \cup \{1\} - \mathbb{G}_\infty, \quad \forall i \neq j \in \mathbb{I}_\theta; \\ \text{torsion:} & q_{ii}, q_{ij}q_{ji} \in \mathbb{G}_\infty, \quad \forall i \neq j \in \mathbb{I}_\theta; \\ \text{semigeneric:} & \text{otherwise.} \end{cases}$$

$\mathfrak{q}$ of **Cartan type**: $\exists$ a generalized Cartan matrix $\mathfrak{a} = (a_{ij})_{i,j\in\mathbb{I}_\theta}$ such that

$$q_{ij}q_{ji} = q_{ii}^{a_{ij}}, \qquad \forall i \neq j \in \mathbb{I}_\theta.$$

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We know: if $\mathfrak{a}$ is of affine type, then $\operatorname{GKdim} \mathcal{B}(V) = \infty$.

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- (Andruskiewitsch–A–Rosso) Assume $\mathfrak{q}$ gen. Then $\operatorname{GKdim} \mathcal{B}(V) < \infty \Leftrightarrow \mathfrak{q}$ of finite Cartan type.

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Conjecture

$\mathfrak{q}$ of Cartan type, $\operatorname{GKdim} \mathcal{B}(V) < \infty \Rightarrow$ finite type.

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$\mathfrak{q}$ torsion, $\operatorname{GKdim} \mathcal{B}(V) < \infty \Rightarrow \dim \mathcal{B}(V) < \infty$.

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Theorem

Last conjecture is true when $\dim V = 2$.

Block $\mathcal{V}(\epsilon, \ell)$: $\exists$ a basis $(x_i)_{i \in \mathbb{I}_\ell}$ such that

$$c(x_i \otimes x_j) = \begin{cases} \epsilon x_1 \otimes x_i, & j = 1 \\ (\epsilon x_j + x_{j-1}) \otimes x_i, & j \geq 2, \end{cases} \qquad i \in \mathbb{I}_\ell.$$

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1. $\operatorname{GKdim} \mathcal{B}(\mathcal{V}(\epsilon, \ell)) < \infty \iff \ell = 2$ and $\epsilon \in \{\pm 1\}$.

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Theorem

1. $\operatorname{GKdim}\mathcal{B}(\mathcal{V}(\epsilon,\ell))<\infty \iff \ell=2$ and $\epsilon\in\{\pm1\}$.
2. $\mathcal{B}(\mathcal{V}(1,2))=\mathbb{C}\langle x_1,x_2|x_2x_1-x_1x_2+\frac{1}{2}x_1^2\rangle$ Jordan plane.
   $\operatorname{GKdim}\mathcal{B}(\mathcal{V}(1,2))=2$, with basis $\{x_1^a x_2^b : a,b\in\mathbb{N}_0\}$.

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   $\operatorname{GKdim} \mathcal{B}(\mathcal{V}(1, 2)) = 2$, with basis $\{x_1^a x_2^b : a, b \in \mathbb{N}_0\}$.
3. $\mathcal{B}(\mathcal{V}(-1, 2)) = \mathbb{C}\langle x_1, x_2 | x_1^2, x_2x_{12} - x_{12}x_2 - x_1x_{12} \rangle$ super Jordan plane. Here $x_{12} = \operatorname{ad}_c x_2 \, x_1 = x_2x_1 + x_1x_2$.
   $\operatorname{GKdim} \mathcal{B}(\mathcal{V}(-1, 2)) = 2$, with basis $\{x_1^a x_{12}^b x_2^c : a \in \{0, 1\}, b, c \in \mathbb{N}_0\}$.

Γ abelian: $V \in {}^{\mathbb{C}\Gamma}_{\mathbb{C}\Gamma}\mathcal{YD} \hookrightarrow V = V_1 \oplus \cdots \oplus V_t \oplus V_{t+1} \oplus \cdots \oplus V_\theta$, where

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$V \in {}^{\mathbb{C}\Gamma}_{\mathbb{C}\Gamma}\mathcal{YD}$, $\dim V = 3$, not of diagonal type

$$\Rightarrow \qquad V = \mathcal{V}_{g_1}(\chi_1, \eta) \oplus \mathbb{C}^{\chi_2}_{g_2}, \qquad g_1, g_2 \in \Gamma,$$

$\chi_1, \chi_2 \in \widehat{\Gamma}$ and $\eta : \Gamma \to \mathbb{C}$ is a (χ_1, χ_1)-derivation.

- $V_1 = \mathcal{V}_{g_1}(\chi_1, \eta) \in {}^{\mathbb{C}\Gamma}_{\mathbb{C}\Gamma}\mathcal{YD}$ is indecomposable with basis $(x_i)_{i \in \mathbb{I}_2}$;

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- $V_2 = \mathbb{C}^{\chi_2}_{g_2} \in {}^{\mathbb{C}\Gamma}_{\mathbb{C}\Gamma}\mathcal{YD}$ is irreducible with base (x_3);
- $\eta(g_1) \neq 0$ (V not diagonal type); may suppose $\eta(g_1) = 1$.

Γ abelian: $V \in {}^{\mathbb{C}\Gamma}_{\mathbb{C}\Gamma}\mathcal{YD} \hookrightarrow V = V_1 \oplus \cdots \oplus V_t \oplus V_{t+1} \oplus \cdots \oplus V_\theta$, where

- $V_1, \dots, V_t \in {}^{\mathbb{C}\Gamma}_{\mathbb{C}\Gamma}\mathcal{YD}$ are blocks;
- $V_{t+1}, \dots, V_\theta \in {}^{\mathbb{C}\Gamma}_{\mathbb{C}\Gamma}\mathcal{YD}$ are points.

$V \in {}^{\mathbb{C}\Gamma}_{\mathbb{C}\Gamma}\mathcal{YD}$, $\dim V = 3$, not of diagonal type

$$\Rightarrow \qquad V = \mathcal{V}_{g_1}(\chi_1, \eta) \oplus \mathbb{C}^{\chi_2}_{g_2}, \qquad g_1, g_2 \in \Gamma,$$

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- Set $q_{ij} = \chi_j(g_i)$, $i, j \in \mathbb{I}_2$; $\epsilon = q_{11}$; $a = q_{21}^{-1}\eta(g_2)$.

- The braiding is given in the basis $(x_i)_{i\in\mathbb{I}_3}$ by

$$(c(x_i \otimes x_j))_{i,j\in\mathbb{I}_3} = \begin{pmatrix} \epsilon x_1 \otimes x_1 & (\epsilon x_2 + x_1) \otimes x_1 & q_{12} x_3 \otimes x_1 \\ \epsilon x_1 \otimes x_2 & (\epsilon x_2 + x_1) \otimes x_2 & q_{12} x_3 \otimes x_2 \\ q_{21} x_1 \otimes x_3 & q_{21}(x_2 + a x_1) \otimes x_3 & q_{22} x_3 \otimes x_3 \end{pmatrix}.$$

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Definition

- $q_{12}q_{21}$ = the **interaction** between the block and the point; it is

  weak if $q_{12}q_{21} = 1$, mild if $q_{12}q_{21} = -1$, strong if $q_{12}q_{21} \notin \{\pm 1\}$.

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  weak if $q_{12}q_{21}=1$, mild if $q_{12}q_{21}=-1$, strong if $q_{12}q_{21}\notin\{\pm1\}$.

- The **ghost**: a normalized version of a,

$$\mathcal{G}=\begin{cases}-2a, & \epsilon=1,\\ a, & \epsilon=-1.\end{cases}$$

  If $\mathcal{G}\in\mathbb{N}$, then we say that the ghost is **discrete**.

Theorem

V b. v. s., braiding as above, $\operatorname{GKdim} \mathcal{B}(V) < \infty \Rightarrow$

interaction	ϵ	q_{22}	$\mathcal{G}$	$\mathcal{B}(V)$	GKdim
weak	± 1	1 or $\notin \mathbb{G}_\infty$	0	$\mathcal{B}(\mathcal{V}(\epsilon, 1)) \underline{\otimes} \mathcal{B}(\mathbb{C}x_3)$	3
		$\in \mathbb{G}_\infty - \{1\}$			2
	1	1	discrete	$\mathcal{B}(\mathfrak{L}(1, 1, \mathcal{G}))$	$\mathcal{G} + 3$
		-1	discrete	$\mathcal{B}(\mathfrak{L}(1, -1, \mathcal{G}))$	2
		$\in \mathbb{G}'_3$	1	$\mathcal{B}(\mathfrak{L}(1, \omega, \mathcal{G}))$	2
	-1	1	discrete	$\mathcal{B}(\mathfrak{L}(-1, 1, \mathcal{G}))$	$\mathcal{G} + 3$
		-1	discrete	$\mathcal{B}(\mathfrak{L}(-1, -1, \mathcal{G}))$	$\mathcal{G} + 2$
mild	-1	-1	1	$\mathcal{B}(\mathfrak{C})$	2

Theorem

V	generators and relations
$\mathfrak{L}(1,1,\mathcal{G})$	$\mathbb{C}\langle x_1, x_2, x_3 \vert x_2x_1 - x_1x_2 + \frac{1}{2}x_1^2, x_1x_3 - q_{12}\, x_3x_1,$ $z_{1+\mathcal{G}}, z_t z_{t+1} - q_{21}q_{22}\, z_{t+1}z_t, 0 \le t < \mathcal{G}\rangle$
$\mathfrak{L}(1,-1,\mathcal{G})$	$\mathbb{C}\langle x_1, x_2, x_3 \vert x_2x_1 - x_1x_2 + \frac{1}{2}x_1^2,$ $x_1x_3 - q_{12}\, x_3x_1, z_{1+\mathcal{G}}, z_t^2, 0 \le t \le \mathcal{G}\rangle$
$\mathfrak{L}(-1,1,\mathcal{G})$	$\mathbb{C}\langle x_1, x_2, x_3 \vert x_1^2, x_2x_{12} - x_{12}x_2 - x_1x_{12},$ $x_1x_3 - q_{12}\, x_3x_1, x_{12}x_3 - q_{12}^2x_3x_{12}, z_{1+2\mathcal{G}},$ $z_{2k+1}^2, z_{2k}z_{2k+1} - q_{21}q_{22}\, z_{2k+1}z_{2k}, 0 \le k < \mathcal{G}\rangle$
$\mathfrak{L}(-1,-1,\mathcal{G})$	$\mathbb{C}\langle x_1, x_2, x_3 \vert x_1^2, x_2x_{12} - x_{12}x_2 - x_1x_{12}, x_3^2,$ $x_1x_3 - q_{12}\, x_3x_1, x_{12}x_3 - q_{12}^2x_3x_{12}, z_{1+2\mathcal{G}},$ $z_{2k}^2, z_{2k-1}z_{2k} - q_{21}q_{22}\, z_{2k}z_{2k-1}, 0 < k \le \mathcal{G}\rangle$
$\mathfrak{L}(1,\omega,1)$	$\mathbb{C}\langle x_1, x_2, x_3 \vert x_2x_1 - x_1x_2 + \frac{1}{2}x_1^2,$ $x_1x_3 - q_{12}\, x_3x_1,\ z_2,\ x_3^3,\ z_1^3,\ z_{1,0}^3\rangle$
$\mathfrak{C}$	$\mathbb{C}\langle x_1, x_2, x_3 \vert x_1^2, x_2x_{12} - x_{12}x_2 - x_1x_{12},$ $x_3^2, f_1^2,\ z_1^2,\ x_{12}x_3 - q_{12}^2x_3x_{12},$ $x_2z_1 + q_{12}z_1x_2 - q_{12}\, f_0x_2 - \frac{1}{2}f_1\rangle$

Here, $z_t = (\operatorname{ad}_c x_2)^t x_3,\ f_t = (\operatorname{ad}_c x_1) z_t.$

Theorem

V b. v. s., braiding as above. Assume Conjecture on diagonal braidings is true. $\operatorname{GKdim} \mathcal{B}(V) < \infty \Rightarrow$

interaction	ϵ	diagram of V_J	$(\mathcal{G}_i, \mathcal{G}_j)$
weak	1	$\overset{\omega}{\underset{i}{\circ}} \overset{\omega^2}{\text{———}} \overset{-1}{\underset{j}{\circ}}$	$(0,1)$
			$(1,0)$
		$\overset{-1}{\underset{i}{\circ}} \overset{-1}{\text{———}} \overset{-1}{\underset{j}{\circ}}$	$(1,0)$
			$(2,0)$
		$\overset{-1}{\underset{i}{\circ}} \overset{\omega}{\text{———}} \overset{-1}{\underset{j}{\circ}}$	$(1,0)$
		$\overset{-1}{\underset{i}{\circ}} \overset{r^{-1}}{\text{———}} \overset{r}{\underset{j}{\circ}}$	$(1,0)$
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Theorem

interaction	ϵ	diagram of V_J	$(\mathcal{G}_i, \mathcal{G}_j)$
mild, $(-1,1)$	-1	$\overset{-1}{\underset{i}{\circ}} \overset{-1}{\text{\_\_\_\_\_}} \overset{-1}{\underset{j}{\circ}}$	$(1,0)$
weak	1	(1,0,...,0)	$\overset{-1}{\circ} \overset{-1}{\text{\_\_\_\_\_}} \overset{-1}{\circ} \cdots\cdots \overset{-1}{\circ} \overset{-1}{\text{\_\_\_\_\_}} \overset{-1}{\circ}$
		(1,0,...,0)	$\overset{-1}{\circ} \overset{\omega}{\text{\_\_\_\_\_}} \overset{\omega^2}{\circ} \overset{\omega}{\text{\_\_\_\_\_}} \overset{\omega^2}{\circ}$
		(1,0,...,0)	$\overset{-1}{\circ} \overset{\omega}{\text{\_\_\_\_\_}} \overset{\omega^2}{\circ} \overset{\omega^2}{\text{\_\_\_\_\_}} \overset{\omega}{\circ}$
mild, $(-1,1,\dots,1)$	-1	(1,0,...,0)	$\overset{-1}{\circ} \overset{-1}{\text{\_\_\_\_\_}} \overset{-1}{\circ} \cdots\cdots \overset{-1}{\circ} \overset{-1}{\text{\_\_\_\_\_}} \overset{-1}{\circ}$

$R = \oplus_{n \geq 0} R^n \in {}^H_H\mathcal{YD}$ graded connected Hopf algebra, $V = R^1$
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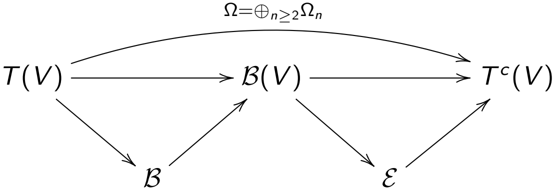
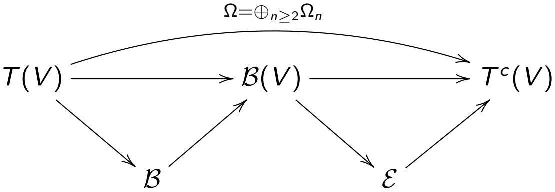
2. R **is coradically graded** $\iff$ $P(R) = V$**:** R is a **post-Nichols** algebra.



$T(V)$ free algebra	$T^c(V)$ free coalgebra	$\mathcal{B}(V)$ Nichols alg.
Ω_n quantum symm.	$\mathcal{B}$ pre-Nichols	$\mathcal{E}$ post-Nichols

Definition

$\mathfrak{Pre}(V)$: poset of pre-Nichols, $\leq$ is $\twoheadrightarrow$; min. $T(V)$, max. $\mathcal{B}(V)$.
$\mathfrak{Post}(V)$: poset of post-Nichols, $\leq$ is $\subseteq$; min. $\mathcal{B}(V)$, max. $T^c(V)$.



$T(V)$ free algebra — Ω_n quantum symm.

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Remark

$\dim V < \infty \Rightarrow \Phi : \mathfrak{Pre}(V) \to \mathfrak{Post}(V^*)$, $\Phi(R) = R^d$, anti-isom.

Let K be a Hopf algebra, $V \in {}^K_K\mathcal{YD}$ finite-dimensional.

Lemma

$\mathcal{B}$ a pre-Nichols algebra of a V, $\mathcal{E} = \mathcal{B}^d$. Then $\operatorname{GKdim} \mathcal{E} \leq \operatorname{GKdim} \mathcal{B}$. If $\mathcal{E}$ is fin. gen., then $\operatorname{GKdim} \mathcal{E} = \operatorname{GKdim} \mathcal{B}$.

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Definition

V is *pre-bounded* if every chain

$$\cdots < \mathcal{B}[3] < \mathcal{B}[2] < \mathcal{B}[1] < \mathcal{B}[0] = \mathcal{B}(V), \tag{1}$$

of pre-Nichols algebras over V with finite GKdim, is finite.

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Lemma

$\mathcal{E} \in {}^K_K\mathcal{YD}$ post-Nichols algebra of V, $\operatorname{GKdim} \mathcal{E} < \infty$. If V^* is pre-bounded, then $\mathcal{E}$ is fin. gen. and $\operatorname{GKdim} \mathcal{E} = \operatorname{GKdim} \mathcal{E}^d$.
In particular, if the only pre-Nichols algebra of V^* with finite GKdim is $\mathcal{B}(V^*)$, then $\mathcal{E} = \mathcal{B}(V)$.

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The only pre-Nichols or post-Nichols algebra of V with finite GKdim is $\mathcal{B}(V)$ when:

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Remark

The only pre-Nichols or post-Nichols algebra of V with finite GKdim is $\mathcal{B}(V)$ when:

- V is of finite generic Cartan type (Andruskiewitsch-Schneider),
- V is of Jordan or super Jordan type.

Theorem

Theorem

1. $\mathcal{D} = (g, \chi, \eta)$ a Jordanian YD-triple for $\mathbb{C}G$, $V = \mathcal{V}_g(\chi, \eta)$. Let $\lambda \in \mathbb{C}$ be such that

$$\lambda = 0, \qquad \text{if } \chi^2 \neq \varepsilon.$$

$\mathfrak{U} = \mathfrak{U}(\mathcal{D}, \lambda) = T(V)\#\mathbb{C}G/\langle x_2x_1 - x_1x_2 + \frac{1}{2}x_1^2 - \lambda(1 - g^2)\rangle$ is a lifting of a Jordan plane. Moreover, every lifting of a Jordan plane is $\mathfrak{U}$ for some data.

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2. $\mathcal{D} = (g, \chi, \eta)$ a super Jordanian YD-triple for $\mathbb{C}G$, $V = \mathcal{V}_g(\chi, \eta)$. Let $\lambda \in \mathbb{C}$ be such that

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$T(V)\#\mathbb{C}G/\langle x_1^2 - \lambda(1 - g^2), x_2x_{12} - x_{12}x_2 - x_1x_{12} + 2\lambda x_2 + \lambda x_1 g^2\rangle$
is a lifting of a super Jordan plane. Moreover, every lifting of a super Jordan plane is $\mathfrak{U}$ for some data.

$\mathfrak{q} = (q_{ij})_{i,j\in\mathbb{I}_\theta} \rightsquigarrow V$. $\mathcal{B}_{\mathfrak{q}} = \mathcal{B}(V) = T(V)/\mathcal{J}_{\mathfrak{q}}$.

$$c_{ij}^{\mathfrak{q}} := -\min\left\{n \in \mathbb{N}_0 : (n+1)_{q_{ii}}(1 - q_{ii}^n q_{ij} q_{ji}) = 0\right\}, \qquad i \neq j.$$

Assume $\dim \mathcal{B}(V) < \infty$.

- $i \in \mathbb{I}$ is a *Cartan vertex* of $\mathfrak{q}$ if $q_{ij} q_{ji} \overset{(\star)}{=} q_{ii}^{c_{ij}^{\mathfrak{q}}}$, for all $j \neq i$.

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- $\alpha = s_{i_1}^{\mathfrak{q}} s_{i_2} \dots s_{i_k}(\alpha_i) \in \Delta_+^{\mathfrak{q}}$ Cartan root of $\mathfrak{q}$ if $i \in \mathbb{I}$ Cartan vertex of $\rho_{i_k} \dots \rho_{i_2}\rho_{i_1}(\mathfrak{q})$.

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Assume $\dim \mathcal{B}(V) < \infty$.

- $i \in \mathbb{I}$ is a *Cartan vertex* of $\mathfrak{q}$ if $q_{ij}q_{ji} \overset{(\star)}{=} q_{ii}^{c_{ij}^{\mathfrak{q}}}$, for all $j \neq i$.
- $\alpha = s_{i_1}^{\mathfrak{q}} s_{i_2} \dots s_{i_k}(\alpha_i) \in \Delta_+^{\mathfrak{q}}$ Cartan root of $\mathfrak{q}$ if $i \in \mathbb{I}$ Cartan vertex of $\rho_{i_k} \dots \rho_{i_2}\rho_{i_1}(\mathfrak{q})$.

Definition

$\mathfrak{G}$ = set of generators of $\mathcal{J}_{\mathfrak{q}}$ [A]. Let $\mathcal{J}_{\mathfrak{q}} \supset \mathcal{I}_{\mathfrak{q}} := \langle \mathfrak{G} \cup \mathfrak{G}_2 - \mathfrak{G}_1 \rangle$

$\mathfrak{G}_1 = \{$powers root vectors $E_\alpha^{N_\alpha}$, α Cartan root$\}$;

$\mathfrak{G}_2 = \{$q Serre rel.$(\operatorname{ad}_c E_i)^{1-c_{ij}^{\mathfrak{q}}} E_j, i \neq j$ such that $1 - c_{ij}^{\mathfrak{q}} = \operatorname{ord} q_{ii}.\}$

The **distinguished pre-Nichols algebra** of V is $\widetilde{\mathcal{B}}_{\mathfrak{q}} = T(V)/\mathcal{I}_{\mathfrak{q}}$.

The *Lusztig algebra* $\mathcal{L}_{\mathfrak{q}}$ of V is the graded dual of $\widetilde{\mathcal{B}}_{\mathfrak{q}}$.

$$\widetilde{\mathcal{B}}_{\mathfrak{q}} \twoheadrightarrow \mathcal{B}_{\mathfrak{q}} \hookrightarrow \mathcal{L}_{\mathfrak{q}}.$$

Theorem

$\operatorname{GKdim} \widetilde{\mathcal{B}}_{\mathfrak{q}} = \operatorname{GKdim}_{\mathfrak{q}} \mathcal{L}_{\mathfrak{q}} = |\{\text{Cartan roots}\}|$.

I. A., *Distinguished pre-Nichols algebras*, Transf. Gr., to appear.

N. Andruskiewitsch, I. A., F. Rossi Bertone. *The quantum divided power algebra of a finite-dimensional Nichols algebra of diagonal type*.

arXiv:1501.04518

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Example

$\mathfrak{q} = (q_{ij})_{i,j\in\mathbb{I}_\theta}$ of Cartan type, i. e. $q_{ij}q_{ji} = q_{ii}^{c_{ij}^{\mathfrak{q}}}$, for all $j \neq i$. Assume $(\operatorname{ord} q_{ij}q_{ji}, 210) = 1$ for all i, j. Then

$$\mathcal{B}_{\mathfrak{q}} = \mathbb{C}\langle x_1, \dots, x_\theta | \text{ quantum Serre rel., power root vectors } \rangle;$$

$$\widetilde{\mathcal{B}}_{\mathfrak{q}} = \mathbb{C}\langle x_1, \dots, x_\theta | \text{ quantum Serre rel. } \rangle$$

$$\operatorname{GKdim} \widetilde{\mathcal{B}}_{\mathfrak{q}} = \operatorname{GKdim}_{\mathfrak{q}} \mathcal{L}_{\mathfrak{q}} = |\Delta^+|.$$