Hopf algebras and Homological Algebra

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Plan

1. Finite injective dimension
2. Finite global dimension
3. Applications
4. Quantum homogeneous spaces

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(1) affine of finite GK-dimension; or
(2) noetherian.
We assume throughout that S is bijective.

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Question

(a) Does (1)$\Rightarrow$ (2)? (b) Does each of (1) or (2) imply S bijective?

1. Finite injective dimension

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That is, H is a Frobenius algebra; in particular, H is an injective (right and left) H-module - we write $\mathrm{injdim} H = 0$.

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2. $\dim_k \mathrm{Ext}^d_A(k, A_{|A}) = 1$; *and* $\mathrm{Ext}^i_A(k, A_{|A}) = 0$ *if* $i \neq d$;
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2. *A a connected graded affine commutative k-algebra, then* $\mathrm{injdim} A < \infty \Leftrightarrow A$ *is AS-Gorenstein; and then* $d = \mathrm{GKdim} A$.
3. *The algebra R of* 2×2 *upper triangular matrices over k has* $\mathrm{injdim} R = 1$, *but R is not AS-Gorenstein.*

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Let A be a noetherian k-algebra.

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2. *Suppose that $\mathrm{GKdim} A < \infty$. Then A is GK-Cohen Macaulay if*

$$j(M) + \mathrm{GKdim} M = \mathrm{GKdim} A$$

for every non-zero f.g. (right or left) A-module M.

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4. *So the "real point" of the GK-Cohen Macaulay condition is to tell us that* $\delta(-)$ *is a symmetric dimension function.*

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2. *First non-trivial general case: (Wu, Zhang, 2003) Noetherian affine PI Hopf algebras are AS-Gorenstein and GK-Cohen Macaulay.*
3. *Second non-trivial general case: (Zhuang, 2013) If H is a connected Hopf algebra of finite GK-dimension, then H is AS-Gorenstein and GK-Cohen Macaulay.*

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For example, what about the case of pointed H?

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2. *Commutative affine Hopf k-algebras (k char 0) are AS-regular.*
3. *Enveloping algebras ($\mathfrak{g}$ f.d.), quantised env. algebras, quantised function algebras are AS-regular.*
4. *Group algebra kG, (G polycyclic-by-finite) is AS-regular* $\Leftrightarrow$ *G has no element of order* $\mathrm{char}\, k$.

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Hence:

Question

Is there an (easily checkable) structural property of a Hopf algebra (assumed affine of finite GKdim or noetherian), which is equivalent to finite global dimension?

3. Applications:(I)Homological

Definition (Lu-Wu-Zhang, 2007)

Let H be an AS-Gorenstein Hopf algebra with $\mathrm{injdim} H = d$. The (left) homological integral $\int_H^\ell$ of H is the 1-dimensional H-bimodule $\mathrm{Ext}^d(k, H)$.

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Theorem (B-Zhang, 2008)

Let H be an AS-Gorenstein Hopf algebra with $\mathrm{injdim} A = d$. Let χ be the character of the right structure on $\int_H^\ell$.

1. *H has a rigid dualising complex $R \cong {}^{\nu}H^1[d]$.*
2. *The Nakayama automorphism ν is $S^2\tau_\chi^\ell$, where τ_χ^ℓ denotes the left winding automorphism of H got from χ.*

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Corollary

Poincaré duality/twisted Calabi-Yau: Let H be an AS-regular Hopf algebra with $\mathrm{injdim} H = d$*. Keep the notation as in the theorem. For every H-bimodule M and for every i,* $H^i(H, {}^{\nu}M^1) \cong H_{d-i}(H, M)$.

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Let H be an AS-Gorenstein Hopf algebra, notation as before. Then, for some inner auto γ of H,

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Question

For H AS-Gorenstein, what is γ in the formula for S^4?

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Similar for H pointed noetherian?

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See [B-Gilmartin, arXiv2015] for the case of H connected.
Also work by a number of people (Kraehmer, Liu, Wu...) on the twisted Calabi-Yau property of various quantum homogeneous spaces.

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Let's look at the case of the Podleś spheres....

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(Following [Muller, Schneider,1999]) Let $\alpha, \beta, \gamma \in k$ *and consider the* $(K^{-1}, 1)$*-primitive element*

$$X := \alpha E K^{-1} + \beta F + \gamma(K^{-1} - 1) \in U_q(\mathfrak{sl}(2)) \setminus \{0\}.$$

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In this case B is called a Podleś quantum sphere. If $(\alpha, \beta, \gamma) = (0, 0, 1)$ *then B is the standard Podleś quantum sphere.*

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Theorem

(Kraehmer, 2012; Liu, Shen, Wu, 2014) Let B be a Podleś quantum sphere.

1. *B is Auslander-regular, GK-Cohen Macaulay and AS-regular of dim. 2.*
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Question

(Kraehmer; Liu, Shen, Wu) Does twisted Calabi-Yau also hold for B when $\alpha\beta \neq 0$?