Reflection Hopf Algebras

Ellen Kirkman

kirkman@wfu.edu

WAKE FOREST UNIVERSITY
Department of Mathematics

Nichols Algebras and their Interaction with Lie Theory, Hopf Algebras and Tensor Categories

BIRS September 7, 2015

Collaborators

- Jacque Alev
- Kenneth Chan
- James Kuzmanovich
- Chelsea Walton
- James Zhang

Setting – Invariant Theory of AS Regular Algebras

$\Bbbk = \mathbb{C}$

Pairs (A, H)

Setting – Invariant Theory of AS Regular Algebras

$\Bbbk = \mathbb{C}$ Pairs (A, H)

- A is an AS regular algebra generated in degree 1

$\Bbbk = \mathbb{C}$ Pairs (A, H)

- A is an AS regular algebra generated in degree 1
- H is a Hopf algebra acting on A:

Setting – Invariant Theory of AS Regular Algebras

$\Bbbk = \mathbb{C}$ Pairs (A, H)

- A is an AS regular algebra generated in degree 1
- H is a Hopf algebra acting on A:
  - H is semi-simple Hopf algebra

$\mathbb{k} = \mathbb{C}$ Pairs (A, H)

- A is an AS regular algebra generated in degree 1
- H is a Hopf algebra acting on A:
  - H is semi-simple Hopf algebra
  - H preserves the grading on A

$\Bbbk = \mathbb{C}$ Pairs (A, H)

- A is an AS regular algebra generated in degree 1
- H is a Hopf algebra acting on A:
  - H is semi-simple Hopf algebra
  - H preserves the grading on A
  - A is an H-module algebra

Setting – Invariant Theory of AS Regular Algebras

$\mathbb{k} = \mathbb{C}$ Pairs (A, H)

- A is an AS regular algebra generated in degree 1
- H is a Hopf algebra acting on A:
  - H is semi-simple Hopf algebra
  - H preserves the grading on A
  - A is an H-module algebra
  - The action of H on A is inner-faithful.

Setting – Invariant Theory of AS Regular Algebras

$\Bbbk = \mathbb{C}$ Pairs (A, H)

- A is an AS regular algebra generated in degree 1
- H is a Hopf algebra acting on A:
  - H is semi-simple Hopf algebra
  - H preserves the grading on A
  - A is an H-module algebra
  - The action of H on A is inner-faithful.
- $A^H = \{a \in A \mid h.a = \epsilon(h)a$ for all $h \in H\}$.

Inner-Faithful Action

Def: H acts inner-faithfully on A if there is no non-zero Hopf ideal I of H with $IA = 0$.

Def: H acts inner-faithfully on A if there is no non-zero Hopf ideal I of H with $IA = 0$.

Proposition: Let

$$B = A_1 \oplus (A_1 \otimes A_1) \oplus (A_1 \otimes A_1 \otimes A_1) \oplus \cdots$$

H acts inner-faithfully on A if and only if every simple H-module appears in the semisimple decomposition of B as an H-module.

Outline

I. Examples of Actions

II. Reflection Groups

III. Reflection Hopf Algebras

IV. Dual Reflection Groups

Survey Paper: arXiv 1506.06121

Examples of Actions: A commutative

Etingof and Walton (2013): If

- A is a commutative domain
- H is a finite dimensional semisimple Hopf algebra over a field of characteristic zero
- A is an H-module algebra for an inner faithful action of H on A,

then H is a group algebra.

Examples of Actions: Actions Must Preserve the Relations of A

Let $A = \mathbb{k}_q[x, y]$ $(yx = qxy)$ and let g be the map that interchanges x and y. For g to define a homomorphism on A need $q = \pm 1$:

Let $A = \mathbb{k}_q[x, y]$ $(yx = qxy)$ and let g be the map that interchanges x and y. For g to define a homomorphism on A need $q = \pm 1$:

$$yx = qxy$$

Wake Forest University
Department of Mathematics

Let $A = \mathbb{k}_q[x, y]$ $(yx = qxy)$ and let g be the map that interchanges x and y. For g to define a homomorphism on A need $q = \pm 1$:

$$yx = qxy$$
$$g(yx) = g(qxy)$$

Let $A = \Bbbk_q[x, y]$ ($yx = qxy$) and let g be the map that interchanges x and y. For g to define a homomorphism on A need $q = \pm 1$:

$$\begin{aligned} yx &= qxy \\ g(yx) &= g(qxy) \\ g(y)g(x) &= qg(x)g(y) \end{aligned}$$

Examples of Actions: Actions Must Preserve the Relations of A

Let $A = \mathbb{k}_q[x, y]$ $(yx = qxy)$ and let g be the map that interchanges x and y. For g to define a homomorphism on A need $q = \pm 1$:

$$\begin{aligned} yx &= qxy \\ g(yx) &= g(qxy) \\ g(y)g(x) &= qg(x)g(y) \\ xy = qyx &= q^2xy. \end{aligned}$$

Examples of Actions: H= Kac-Paljutkin Hopf algebra

Kac-Paljutkin Hopf algebra H_8

Find algebras A on which H_8 acts inner faithfully

Kac-Paljutkin Hopf algebra H_8

Find algebras A on which H_8 acts inner faithfully
If A is AS regular of dimension 2

$$A = \Bbbk\langle u, v\rangle/(r).$$

Kac-Paljutkin Hopf algebra H_8

Find algebras A on which H_8 acts inner faithfully
If A is AS regular of dimension 2

$$A = \Bbbk\langle u, v\rangle/(r).$$

H_8 has a unique two-dimensional representation S and 4 one-dimensional representations T_i.

Kac-Paljutkin Hopf algebra H_8

Find algebras A on which H_8 acts inner faithfully
If A is AS regular of dimension 2

$$A = \Bbbk\langle u, v\rangle/(r).$$

H_8 has a unique two-dimensional representation S and 4 one-dimensional representations T_i.

Let ${}_{H_8}A_1 \cong S$

Kac-Paljutkin Hopf algebra H_8

Find algebras A on which H_8 acts inner faithfully
If A is AS regular of dimension 2

$$A = \Bbbk\langle u, v\rangle/(r).$$

H_8 has a unique two-dimensional representation S and 4 one-dimensional representations T_i.

Let ${}_{H_8}A_1 \cong S$ and ${}_{H_8}\Bbbk r \cong T_i$ where T_i appears in $S \otimes S \cong T_1 \oplus T_2 \oplus T_3 \oplus T_4$. Take $\langle r\rangle = T_i$.

Invariants of H_8 acting on $A = \Bbbk\langle u, v\rangle/(r)$.

Relation r	Fixed Ring A^{H_8}
$u^2 + v^2$	comm complete intersection
$u^2 - v^2$	comm poly $\Bbbk[u^2, (uv)^2 - (vu)^2]$
$uv + ivu$	comm poly $\Bbbk[u^2 + v^2, u^2v^2]$
$uv - ivu$	comm poly $\Bbbk[u^2 + v^2, u^2v^2]$

Examples of Actions: Start with A
Generalize finite subgroups and invariants of $SL_2(\mathbb{C})$

Actions of Quantum Binary Polyhedral Groups on Quantum Planes [Chan, K, Walton, Zhang]

Examples of Actions: Start with A
Generalize finite subgroups and invariants of $SL_2(\mathbb{C})$

Actions of Quantum Binary Polyhedral Groups on Quantum Planes [Chan, K, Walton, Zhang]

Find all H, a f.d. Hopf algebra acting on A, an AS regular algebra of dim 2:

$$\Bbbk_J[u,v] := \Bbbk\langle u,v\rangle/(vu - uv - u^2)$$

or $$\Bbbk_q[u,v] := \Bbbk\langle u,v\rangle/(vu - quv),$$

with trivial *homological determinant*, so that A is an H module algebra, the action is inner faithful and preserves the grading.

Examples of Actions: Start with A
Generalize finite subgroups and invariants of $SL_2(\mathbb{C})$

Actions of Quantum Binary Polyhedral Groups
on Quantum Planes [Chan, K, Walton, Zhang]

Find all H, a f.d. Hopf algebra acting on A,
an AS regular algebra of dim 2:

$$\Bbbk_J[u,v] := \Bbbk\langle u,v\rangle/(vu - uv - u^2)$$

or $$\Bbbk_q[u,v] := \Bbbk\langle u,v\rangle/(vu - quv),$$

with trivial *homological determinant*, so that A is an H module algebra, the action is inner faithful and preserves the grading.

Use the classification of finite Hopf quotients of the coordinate Hopf algebra $\mathcal{O}_q(SL_2(\Bbbk))$ (Bichon-Natale, Müller, Stefan).

WAKE FOREST UNIVERSITY
Department of Mathematics

Examples of Actions:
Quantum Binary Polyhedral Groups on Quantum Planes

AS reg alg A gldim 2	f.d. Hopf algebra(s) H acting on A
$\Bbbk[u,v]$	$\Bbbk\tilde{\Gamma}$
$\Bbbk_{-1}[u,v]$	$\Bbbk C_n$ for $n \geq 2$; $\Bbbk D_{2n}$; $(\Bbbk D_{2n})^\circ$; $\mathcal{D}(\tilde{\Gamma})^\circ$ for $\tilde{\Gamma}$ nonabelian
$\Bbbk_q[u,v]$, q root of 1, $q^2 \neq 1$	
if U non-simple	$\Bbbk C_n$ for $n \geq 3$; $(T_{q,\alpha,n})^\circ$;
if U simple, $o(q)$ odd	$1 \to (\Bbbk\tilde{\Gamma})^\circ \to H^\circ \to \mathfrak{u}_q(\mathfrak{sl}_2)^\circ \to 1$;
if U simple, $o(q)$ even, and $q^4 \neq 1$	$1 \to (\Bbbk\Gamma)^\circ \to H^\circ \to \mathfrak{u}_{2,q}(\mathfrak{sl}_2)^\circ \to 1$;
if U simple, $q^4 = 1$	$1 \to (\Bbbk\Gamma)^\circ \to H^\circ \to \mathfrak{u}_{2,q}(\mathfrak{sl}_2)^\circ \to 1$ $1 \to (\Bbbk\Gamma)^\circ \to H^\circ \to \frac{\mathfrak{u}_{2,q}(\mathfrak{sl}_2)^\circ}{(e_{12}-e_{21}e_{11}^2)} \to 1$
$\Bbbk_q[u,v]$, q not root 1	$\Bbbk C_n, n \geq 2$
$\Bbbk_J[u,v]$	$\Bbbk C_2$

WAKE FOREST UNIVERSITY
Department of Mathematics

Let $\Bbbk$ be a field of characteristic zero.

Theorem (1954). The ring of invariants $\Bbbk[x_1, \cdots, x_n]^G$ under a finite group G is a polynomial ring if and only if G is a reflection group (i.e. generated by reflections).

Let $\mathbb{k}$ be a field of characteristic zero.

Theorem (1954). The ring of invariants $\mathbb{k}[x_1, \cdots, x_n]^G$ under a finite group G is a polynomial ring if and only if G is a reflection group (i.e. generated by reflections).

A linear map g on V is called a reflection of V if all but one of the eigenvalues of g are 1, i.e. dim $V^g =$ dim $V - 1$.

Coinvariant ring:

$$\frac{\Bbbk[x_1, \ldots, x_n]}{(f_1, \ldots, f_n)}$$

Coinvariant ring:

$$\frac{\Bbbk[x_1, \ldots, x_n]}{(f_1, \ldots, f_n)}$$

is a complete intersection,

Coinvariant ring:

$$\frac{\Bbbk[x_1, \ldots, x_n]}{(f_1, \ldots, f_n)}$$

is a complete intersection, and lies in a Nichols algebra $\mathcal{B}(M_G)$ (Fomin-Kirillov, Milinski-Schneider, Bazlov, Kirillov-Maeno)

Noncommutative Gauss' Theorem?

Noncommutative Gauss' Theorem?

Example: $S_2 = \langle g \rangle$,

Noncommutative Gauss' Theorem?

Example: $S_2 = \langle g \rangle$, for $g = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, acts on

$A = \mathbb{k}_{-1}[u, v]$

Noncommutative Gauss' Theorem?

Example: $S_2 = \langle g \rangle$, for $g = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, acts on $A = \mathbb{k}_{-1}[u, v]$ and A^{S_2} is generated by

$$P_1 = u + v \text{ and } P_2 = u^3 + v^3$$

Noncommutative Gauss' Theorem?

Example: $S_2 = \langle g \rangle$, for $g = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, acts on $A = \Bbbk_{-1}[u, v]$ and A^{S_2} is generated by

$$P_1 = u + v \text{ and } P_2 = u^3 + v^3$$

$(u^2 + v^2 = (u + v)^2$

Noncommutative Gauss' Theorem?

Example: $S_2 = \langle g \rangle$, for $g = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, acts on $A = \Bbbk_{-1}[u, v]$ and A^{S_2} is generated by

$$P_1 = u + v \text{ and } P_2 = u^3 + v^3$$

$(u^2 + v^2 = (u + v)^2$ and $g \cdot uv = vu = -uv$ so no generators in degree 2).

Noncommutative Gauss' Theorem?

Example: $S_2 = \langle g \rangle$, for $g = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, acts on $A = \mathbb{k}_{-1}[u, v]$ and A^{S_2} is generated by

$$P_1 = u + v \text{ and } P_2 = u^3 + v^3$$

$(u^2 + v^2 = (u + v)^2$ and $g \cdot uv = vu = -uv$ so no generators in degree 2).The generators are NOT algebraically independent.

Noncommutative Gauss' Theorem?

Example: $S_2 = \langle g \rangle$, for $g = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, acts on $A = \Bbbk_{-1}[u, v]$ and A^{S_2} is generated by

$$P_1 = u + v \text{ and } P_2 = u^3 + v^3$$

$(u^2 + v^2 = (u+v)^2$ and $g \cdot uv = vu = -uv$ so no generators in degree 2). The generators are NOT algebraically independent.

A^{S_2} is NOT AS-regular (but it is a hyperplane in an AS-regular algebra).

Noncommutative Gauss' Theorem?

Example: $S_2 = \langle g \rangle$, for $g = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, acts on $A = \mathbb{k}_{-1}[u, v]$ and A^{S_2} is generated by

$$P_1 = u + v \text{ and } P_2 = u^3 + v^3$$

$(u^2 + v^2 = (u + v)^2$ and $g \cdot uv = vu = -uv$ so no generators in degree 2). The generators are NOT algebraically independent.

A^{S_2} is NOT AS-regular (but it is a hyperplane in an AS-regular algebra).

The transposition $(1, 2)$ is NOT a "reflection".

Reflection Group: What is a reflection?

Definition of “reflection": Want A^G AS regular

Reflection Group: What is a reflection?

Definition of “reflection": Want A^G AS regular

All but one eigenvalue of g is 1 $\rightsquigarrow$

Definition of “reflection": Want A^G AS regular

All but one eigenvalue of g is 1 $\rightsquigarrow$

The trace function of g acting on A of dimension n has a pole of order $n-1$ at $t=1$, where

$$Tr_A(g,t) = \sum_{k=0}^{\infty} trace(g|A_k)t^k$$

$$= \frac{1}{(1-t)^{n-1}q(t)} \text{ for } q(1) \neq 0.$$

Reflection Groups: Examples illustrating the definition of reflection.

$$G = < g > \text{ on } A = \mathbb{k}_{-1}[u, v]$$

Wake Forest University
Department of Mathematics

Reflection Groups: Examples illustrating the definition of reflection.

$G = < g >$ on $A = \Bbbk_{-1}[u, v]$ $(vu = -uv)$:

(a) $g = \begin{bmatrix} \epsilon_n & 0 \\ 0 & 1 \end{bmatrix}, Tr_A(g, t) = \dfrac{1}{(1-t)(1-\epsilon_n t)}$,

$A^g = \Bbbk\langle u^n, v\rangle$ is AS regular (classical reflection).

Reflection Groups: Examples illustrating the definition of reflection.

$G =< g >$ on $A = \Bbbk_{-1}[u, v](vu = -uv)$:

(a) $g = \begin{bmatrix} \epsilon_n & 0 \\ 0 & 1 \end{bmatrix}, Tr_A(g, t) = \dfrac{1}{(1-t)(1-\epsilon_n t)}$,

$A^g = \Bbbk\langle u^n, v\rangle$ is AS regular (classical reflection).

(b) $g = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, Tr_A(g, t) = \dfrac{1}{1+t^2}$,

A^g is NOT AS regular.

$G =< g >$ on $A = \Bbbk_{-1}[u, v](vu = -uv)$:

(a) $g = \begin{bmatrix} \epsilon_n & 0 \\ 0 & 1 \end{bmatrix}, Tr_A(g,t) = \dfrac{1}{(1-t)(1-\epsilon_n t)}$,
$A^g = \Bbbk\langle u^n, v\rangle$ is AS regular (classical reflection).

(b) $g = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, Tr_A(g,t) = \dfrac{1}{1+t^2}$,
A^g is NOT AS regular.

(c) $g = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, Tr_A(g,t) = \dfrac{1}{(1-t)(1+t)}$,
$A^g = \Bbbk[u^2 + v^2, uv]$ is AS regular ("mystic reflection").

Example: The "binary dihedral groups" of order 4ℓ generated by

$$g_1 = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \text{ and } g_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

for λ a primitive 2ℓth root of unity, acts on $A = \mathbb{k}_{-1}[x, y]$.

$$A^G = \mathbb{k}[xy,\ x^{2\ell} + y^{2\ell}].$$

Example: The "binary dihedral groups" of order 4ℓ generated by

$$g_1 = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \text{ and } g_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

for λ a primitive 2ℓth root of unity, acts on $A = \mathbb{k}_{-1}[x, y]$.

$$A^G = \mathbb{k}[xy,\ x^{2\ell} + y^{2\ell}].$$

Question: Is there some Nichols algebra associated to its coinvariants?

Some AS regular algebras have no reflections.

- $\Bbbk_J[x,y] = \Bbbk\langle x,y\rangle/(yx - xy - x^2)$.
- AS regular algebras of dimension 3 with 2 generators.
- Non-PI Sklyanin algebra of dimension 3.
- Sklyanin algebra of dimension 4.
- Homogenization $H(\mathfrak{g})$ of a finite dimensional enveloping algebra with no one dimensional Lie ideal.

Some AS regular algebras have no reflections.

- $\Bbbk_J[x, y] = \Bbbk\langle x, y\rangle/(yx - xy - x^2)$.
- AS regular algebras of dimension 3 with 2 generators.
- Non-PI Sklyanin algebra of dimension 3.
- Sklyanin algebra of dimension 4.
- Homogenization $H(\mathfrak{g})$ of a finite dimensional enveloping algebra with no one dimensional Lie ideal.

Question: Are there AS regular algebras with other kinds of reflections?

[K, Kuzmanovich, Zhang]
G is a reflection group for $A = \Bbbk_{q_{ij}}[x_1, \cdots, x_n]$ iff
G is generated by "reflections".

Shephard-Todd-Chevalley Theorem: H=kG A = skew polynomials

[K, Kuzmanovich, Zhang]
G is a reflection group for $A = \Bbbk_{q_{ij}}[x_1, \cdots, x_n]$ iff G is generated by “reflections”.

[Bazlov and Berenstein] (“Demystification")
If G is a reflection group for $A = \Bbbk_{q_{ij}}[x_1, \cdots, x_n]$, then there is a classical reflection group G' with $\Bbbk G \cong \Bbbk G'$ as algebras.

Conjecture: Generalization of Shephard-Todd-Chevalley Theorem:

A^G is AS regular if and only if G is generated by “reflections".

We call H a reflection Hopf algebra for A if H has an action on A so that A^H is AS regular, $A^H = \{a \in A \mid h.a = \epsilon(h)a$ for all $h \in H\}$.

We call H a reflection Hopf algebra for A if H has an action on A so that A^H is AS regular, $A^H = \{a \in A \mid h.a = \epsilon(h)a$ for all $h \in H\}$.

Examples:

(a)$(\Bbbk G, \Bbbk[x_1, \ldots, x_n])$ for classical reflection group G, or $(\Bbbk G, \Bbbk_{-1}[x_1, \ldots, x_n])$ for mystic reflection group G.

We call H a reflection Hopf algebra for A if H has an action on A so that A^H is AS regular, $A^H = \{a \in A \mid h.a = \epsilon(h)a$ for all $h \in H\}$.

Examples:

(a)$(\Bbbk G, \Bbbk[x_1, \ldots, x_n])$ for classical reflection group G, or $(\Bbbk G, \Bbbk_{-1}[x_1, \ldots, x_n])$ for mystic reflection group G.

(b) H_8 is a reflection Hopf algebra for $\Bbbk_{-1}[x, y]$ and $\Bbbk_{\pm i}[x, y]$.

We call H a <u>reflection Hopf algebra</u> for A if H has an action on A so that A^H is AS regular, $A^H = \{a \in A \mid h.a = \epsilon(h)a$ for all $h \in H\}$.

Examples:

(a)$(\Bbbk G, \Bbbk[x_1, \ldots, x_n])$ for classical reflection group G, or $(\Bbbk G, \Bbbk_{-1}[x_1, \ldots, x_n])$ for mystic reflection group G.

(b) H_8 is a reflection Hopf algebra for $\Bbbk_{-1}[x, y]$ and $\Bbbk_{\pm i}[x, y]$.

Problem: Find all (H, A) so that H is a reflection Hopf algebra for A.

H	A	H as algebra
$H_8 = B_8$	$\Bbbk_{-1}[u,v], \Bbbk_{\pm i}[u,v]$	$\Bbbk D_8$
H_{8n^2}	$\Bbbk_{\pm i}[u,v]$	$\Bbbk G(2n,1,2)$
A_{4m} (m odd)	$\Bbbk_{-1}[u,v], \Bbbk[u,w][v;\sigma]$	$\Bbbk[D_{12}]$
B_{4m}	$\Bbbk_{-1}[u,v]$	$\Bbbk[D_{12}]$

Question: Is a reflection Hopf algebra H always isomorphic (as an algebra) to the $\Bbbk G$ for G a reflection group?

Wake Forest University
Department of Mathematics

These algebras are Hopf extensions:

A_{4m} and B_{4m}:

$$\Bbbk \to k^{\mathbb{Z}_2} \to H \to \Bbbk D_{2m} \to \Bbbk$$

H_{8n^2}:

$$\Bbbk \to \Bbbk^{\mathbb{Z}_{2n} \times \mathbb{Z}_{2n}} \to H \to \Bbbk \mathbb{Z}_2 \to \Bbbk$$

Dual reflection group

$H = \Bbbk G^{\circ} = \Bbbk^{G}$ is a commutative algebra

Wake Forest University
Department of Mathematics

Dual reflection group

$H = \Bbbk G^{\circ} = \Bbbk^{G}$ is a commutative algebra

$S_g \otimes S_h \cong S_{gh}$

Dual reflection group

$H = \Bbbk G^\circ = \Bbbk^G$ is a commutative algebra

$S_g \otimes S_h \cong S_{gh}$

H acts inner faithfully on A if and only if
$A_1 = S_{g_1} \oplus \cdots \oplus S_{g_n}$ where $\{g_i\}$ generate G.

Dual reflection group

$H = \Bbbk G^\circ = \Bbbk^G$ is a commutative algebra

$S_g \otimes S_h \cong S_{gh}$

H acts inner faithfully on A if and only if $A_1 = S_{g_1} \oplus \cdots \oplus S_{g_n}$ where $\{g_i\}$ generate G.

A is graded by G and $A^H = A_e$.

Dual reflection group

$H = \Bbbk G^\circ = \Bbbk^G$ is a commutative algebra

$S_g \otimes S_h \cong S_{gh}$

H acts inner faithfully on A if and only if $A_1 = S_{g_1} \oplus \cdots \oplus S_{g_n}$ where $\{g_i\}$ generate G.

A is graded by G and $A^H = A_e$.

When A_e is AS regular we call G a *dual reflection group for* A.

Wake Forest University
Department of Mathematics

Example: $G = D_8$ is a dual reflection group for $A = \Bbbk_{\pm 1}[u, w][v; \sigma]$, where $\sigma(u) = aw$, $\sigma(w) = bu$,

Example: $G = D_8$ is a dual reflection group for $A = \Bbbk_{\pm 1}[u, w][v; \sigma]$, where $\sigma(u) = aw$, $\sigma(w) = bu$, and D_8 grading $u \mapsto r, v \mapsto r\rho, w \mapsto r\rho^2$.

Example: $G = D_8$ is a dual reflection group for $A = \Bbbk_{\pm 1}[u, w][v; \sigma]$, where $\sigma(u) = aw$, $\sigma(w) = bu$, and D_8 grading $u \mapsto r, v \mapsto r\rho, w \mapsto r\rho^2$.

Relations in A:

$$wu = \pm uw \quad \text{grade } \rho^2$$

$$vu = awv \quad \text{grade } \rho^3$$

$$vw = buv; \quad \text{grade } \rho$$

Example: $G = D_8$ is a dual reflection group for $A = \Bbbk_{\pm 1}[u, w][v; \sigma]$, where $\sigma(u) = aw$, $\sigma(w) = bu$, and D_8 grading $u \mapsto r, v \mapsto r\rho, w \mapsto r\rho^2$.

Relations in A:

$$wu = \pm uw \quad \text{grade } \rho^2$$

$$vu = awv \quad \text{grade } \rho^3$$

$$vw = buv; \quad \text{grade } \rho$$

The invariant subring is

$$A^H = A_e \cong \Bbbk[u^2, w^2][v^2; \sigma']$$

an AS regular algebra.

Dual reflection groups: Examples

Semi-direct product $U \rtimes_\phi V$:

$$U \rtimes_\phi V = \{uv : u \in U, v \in V\}$$

Product: $(u_1v_1)(u_2v_2) = (u_1\phi_{v_1}(u_2))(v_1v_2)$.

Dual reflection groups: Examples

Semi-direct product $U \rtimes_\phi V$:

$$U \rtimes_\phi V = \{uv : u \in U, v \in V\}$$

Product: $(u_1v_1)(u_2v_2) = (u_1\phi_{v_1}(u_2))(v_1v_2)$.

$W \cong W_i = \langle w_i \rangle$ cyclic group order r
$U = W_1 \times \cdots \times W_n$
$V = \langle v \rangle$ cyclic group order np
$\phi : V \longrightarrow S_n$ be $\phi(v) = (1, 2, \ldots, n)$

$$G = U \rtimes_\phi V$$

.

Fix $q := \pm 1$ and $R := \Bbbk_q[x_1, x_2, \dots, x_n]$
$A := R[y; \tau]$, where $\tau(x_i) := x_{i+1}$ for
$i = 1, 2, \dots, n-1$ and $\tau(x_n) := x_1$.

Then A is G-graded by $x_i \mapsto w_i$ and $y \mapsto v$.
The invariant algebra is given by
$A^{k^G} = \Bbbk_Q[x_1^r, x_2^r, \dots, x_n^r][y^{np}]$ where $Q = q^{r^2}$, an
AS regular algebra.

Dual reflection groups: Necessary conditions

Let A be noetherian AS regular domain with $A^H = A_e$ AS regular then:

Dual reflection groups: Necessary conditions

Let A be noetherian AS regular domain with $A^H = A_e$ AS regular then:

- A_g is free of rank 1 over $A^H = A_e$.

Dual reflection groups: Necessary conditions

Let A be noetherian AS regular domain with $A^H = A_e$ AS regular then:

- A_g is free of rank 1 over $A^H = A_e$.
- $A_1 = \Bbbk x_1 \oplus \Bbbk x_2 \oplus \cdots \oplus \Bbbk x_\ell \oplus \Bbbk y_1 \oplus \cdots \oplus \Bbbk y_m$. where $\Bbbk x_i \cong S_{g_i}$ for some $g_i \neq e$ and each $\Bbbk y_j \cong S_e$, where $g_1, g_2, \ldots, g_\ell$ are distinct.

Let A be noetherian AS regular domain with $A^H = A_e$ AS regular then:

- A_g is free of rank 1 over $A^H = A_e$.
- $A_1 = \Bbbk x_1 \oplus \Bbbk x_2 \oplus \cdots \oplus \Bbbk x_\ell \oplus \Bbbk y_1 \oplus \cdots \oplus \Bbbk y_m$. where $\Bbbk x_i \cong S_{g_i}$ for some $g_i \neq e$ and each $\Bbbk y_j \cong S_e$, where $g_1, g_2, \ldots, g_\ell$ are distinct. Inner faithful action implies $\{g_1, g_2, \ldots, g_\ell\}$ generates G.

- $A_g = u_g A^H = A^H u_g$ where u_g is a product $u_g = x_{j_1} x_{j_2} \cdots x_{j_{n_g}}$ and $g = g_{j_1} g_{j_2} \cdots g_{j_{n_g}}$ is the shortest length representation of g in terms of the generating set $\{g_1, g_2, \ldots, g_\ell\}$.

  (n_g is the length of the shortest path from e to g in the Cayley graph for G corresponding to this generating set.)

Dual reflection groups: Necessary conditions

$$A_1 = kx_1 \oplus kx_2 \oplus \cdots \oplus kx_\ell \oplus ky_1 \oplus \cdots ky_m$$

$$A = A^H \oplus \oplus_{g \neq e} u_g A^H$$

Let $p(t) := 1 + \sum_{g \neq e} t^{\ell(g)}$ (the Poincaré polynomial)

$$H_{A^H}(t) = \frac{H_A(t)}{p(t)} \text{ so } p(t)$$

is a product of cyclotomic polynomials.

Dual reflection groups; Necessary conditions

$$p(t) = 1 + \sum_{g \neq e} t^{\ell(g)}, \qquad H_{A^H}(t) = \frac{H_A(t)}{p(t)}.$$

$p(1) = |G|$ and $p(t) = \Phi_{n_1}(t) \cdots \Phi_{n_r}(t)$

$$\Phi_n(1) = \begin{cases} p & n = p^r \text{ for } p \text{ prime} \\ 1 & \text{otherwise} \end{cases}$$

Dual reflection groups: Recall D_8 example

For $H = k^{D_8}$ acting on $A = k\langle u, \quad v, \quad w\rangle$
(group grades) $\qquad r \quad r\rho \quad r\rho^2$

Wake Forest University
Department of Mathematics

For $H = k^{D_8}$ acting on $A = k\langle u, \quad v, \quad w\rangle$
(group grades) $\qquad r \quad r\rho \quad r\rho^2$

(A has PBW basis of the form $u^i w^j v^k$, $B := A^H$)

$$A = B \oplus uB \oplus vB \oplus wB \oplus uvB \oplus uwB \oplus wvB \oplus uwvB$$

$e \quad r \quad r\rho \quad r\rho^2 \quad \rho \quad \rho^2 \quad \rho^3 \quad r\rho^3$

Dual reflection groups: Recall D_8 example

For $H = k^{D_8}$ acting on $A = k\langle u, \quad v, \quad w\rangle$
(group grades) $\quad r \quad r\rho \quad r\rho^2$

(A has PBW basis of the form $u^i w^j v^k$, $B := A^H$)

$$A = B \oplus uB \oplus vB \oplus wB \oplus uvB \oplus uwB \oplus wvB \oplus uwvB$$
$$e \quad r \quad r\rho \quad r\rho^2 \quad \rho \quad \rho^2 \quad \rho^3 \quad r\rho^3$$

$$p(t) = 1 + 3t + 3t^2 + t^3 = (1+t)^3$$

$$p(1) = 8$$

$$H_B(t) = \frac{1}{(1-t)^3(1+t)^3} = \frac{1}{(1-t^2)^3}$$

WAKE FOREST UNIVERSITY
Department of Mathematics

D_{2p} for p prime is not a dual reflection group for ANY quadratic AS regular algebra:

Maximum diameter of Cayley graph is p, so

$p(1) = 2p \quad \Rightarrow$

$$p(t) = \Phi_2(t)\Phi_p(t) = (1+t)(1+t+t^2+\cdots+t^{p-1})$$

$$= 1 + 2t + 2t^2 + \cdots + 2t^{p-1} + t^p$$

$$A = k\langle x_1, \quad x_2, \quad y_1, \ldots, y_m\rangle$$

$$r \quad r\rho, \quad e \quad e$$

Dual reflection groups: Using the necessary conditions

$$A = \Bbbk\langle x_1, \quad x_2, \quad y_1, \ldots, y_m \rangle$$

$$\quad r \quad r\rho, \quad e \quad\quad e$$

Let $p = 3$. If $B = A^H = A_e$ is AS regular

$$A = B \oplus x_1 B \oplus x_2 B \oplus x_1 x_2 B \oplus x_2 x_1 B \oplus x_1 x_2 x_1 B$$

$$e \quad r \quad r\rho \quad \rho \quad \rho^2 \quad r\rho^2$$

$x_2 x_1 x_2$ also has group grade $r\rho^2$, so in A we have $x_2 x_1 x_2 = \lambda x_1 x_2 x_1$, and we show this leads to a contradiction.

Hence D_6 is NOT a dual reflection group for ANY quadratic AS regular A.

WAKE FOREST UNIVERSITY
Department of Mathematics

Problem:

Classify the dual reflection groups G for A.

Given V = generating set of group G,

Hasse algebra $\mathcal{H}_G(V)$ of G wrt V

$\Bbbk$-linear basis G

multiplication determined by

$$g \cdot h = \begin{cases} gh & \ell(gh) = \ell(g) + \ell(h), \\ 0 & \ell(gh) < \ell(g) + \ell(h), \end{cases}$$

Given V = generating set of group G,

Hasse algebra $\mathcal{H}_G(V)$ of G wrt V

$\mathbb{k}$-linear basis G

multiplication determined by

$$g \cdot h = \begin{cases} gh & \ell(gh) = \ell(g) + \ell(h), \\ 0 & \ell(gh) < \ell(g) + \ell(h), \end{cases}$$

(Fomin-Stanley: *Nil-Coxeter Algebra*)

Given V = generating set of group G,

Hasse algebra $\mathcal{H}_G(V)$ of G wrt V

$\Bbbk$-linear basis G

multiplication determined by

$$g \cdot h = \begin{cases} gh & \ell(gh) = \ell(g) + \ell(h), \\ 0 & \ell(gh) < \ell(g) + \ell(h), \end{cases}$$

(Fomin-Stanley: *Nil-Coxeter Algebra*)
(those associated to complex reflection groups lie in a Nichols algebra)

The Poincaré polynomial $p(t)$ is the Hilbert series of the Hasse algebra.

Example: Let $G = D_6$, $V = \{r, r\rho\}$.

$$\mathcal{H}_G(V) = \frac{\Bbbk\langle a, b\rangle}{(a^2, b^2, aba - bab)}$$

$$p(t) = 1 + 2t + 2t^2 + t^3.$$

Theorem [K, Kuzmanovich, Zhang]. When $p(t)$ is a palindrome polynomial, $\mathcal{H}_G(V)$ is a Frobenius algebra, and its graded Nakayama automorphism is given by conjugation by the unique element of maximal length, and preserves length.

Example: $G = D_6$ and $V = \{r, r\rho\}$, maximal length element $r\rho^2$.

Theorem [K, Kuzmanovich, Zhang] If G is a dual reflection group for a noetherian AS regular domain A generated in degree 1, then

- the coinvariant ring $A/((A_e)_{\geq 1})$ is Frobenius
- the basis vector x_m in grade m, where m is the unique element of G of maximal length, is a homogeneous normal element of A
- the conjugation isomorphism η_m of A that maps $a \to x_m a (x_m)^{-1}$ maps $x_g \to \beta(g) x_{m^{-1}gm}$, and so induces the graded Nakayama automorphism of the coinvariant algebra.

Example: $H = k^{D_8}$ and $A = \Bbbk[u, w][v; \sigma]$.

$r.\ r\rho^2, \quad r\rho$

$p(t) = 1 + 3t + 3t^2 + t^3$. $m = r\rho^3$, $x_m = uwv$.

$A^H = \Bbbk[u^2, w^2][v^2; \sigma']$

coinvariants = $\mathcal{H}_V(G) =$
$\Bbbk\langle a, b, c\rangle/(a^2, b^2, c^c, ab - bc, ba - cb, ac - ca)$

Let G be a nonabelian group of order pq, for $p, q,$ primes $p > q \geq 3$. Suppose that the maximal length of any $g \in G$ with respect to any generating set is $\leq p + q$.

Claim: Then G is not a dual reflection group for any noetherian AS regular domain.

$$p(t) = \Phi_p(t)\Phi_q(t)\Phi_6(t) =$$
$$(1 + t + \cdots + t^{p-1})(1 + t + \cdots + t^{q-1})(1 - t + t^2) =$$
$$1 + t + \cdots + t^{p+q-1} + t^{p+q},$$

$$p(t) = \Phi_p(t)\Phi_q(t)\Phi_6(t) =$$
$$(1 + t + \cdots + t^{p-1})(1 + t + \cdots + t^{q-1})(1 - t + t^2) =$$
$$1 + t + \cdots + t^{p+q-1} + t^{p+q},$$

but G must have more than one generator, so

$$p(t) = \Phi_p(t)\Phi_q(t) = 1 + 2t + \cdots + 2t^{p+q-3} + t^{p+q-2}$$

$$p(t) = \Phi_p(t)\Phi_q(t)\Phi_6(t) =$$
$$(1 + t + \cdots + t^{p-1})(1 + t + \cdots + t^{q-1})(1 - t + t^2) =$$
$$1 + t + \cdots + t^{p+q-1} + t^{p+q},$$

but G must have more than one generator, so

$$p(t) = \Phi_p(t)\Phi_q(t) = 1 + 2t + \cdots + 2t^{p+q-3} + t^{p+q-2}$$

and the generating set has exactly 2 elements, which must be permuted under conjugation by m.

$$p(t) = \Phi_p(t)\Phi_q(t)\Phi_6(t) =$$
$$(1 + t + \cdots + t^{p-1})(1 + t + \cdots + t^{q-1})(1 - t + t^2) =$$
$$1 + t + \cdots + t^{p+q-1} + t^{p+q},$$

but G must have more than one generator, so

$$p(t) = \Phi_p(t)\Phi_q(t) = 1 + 2t + \cdots + 2t^{p+q-3} + t^{p+q-2}$$

and the generating set has exactly 2 elements, which must be permuted under conjugation by m. So either conjugation by m is the identity (so that m is central, a contradiction) or it has order 2. But then 2 must divide the order of the group, a contradiction.

Questions:

- For what groups G is $H = k^G$ a reflection Hopf algebra for an AS regular algebra A?

Questions:

- For what groups G is $H = k^G$ a reflection Hopf algebra for an AS regular algebra A?
- When is H (H°) a reflection Hopf algebra for an AS regular algebra A?

Questions:

- For what groups G is $H = k^G$ a reflection Hopf algebra for an AS regular algebra A?
- When is H (H°) a reflection Hopf algebra for an AS regular algebra A?
- For H a reflection Hopf for A is $A/(A^H)_{\geq 1}$ Frobenius, complete intersection, of dim(H), related to a Nichols algebra??

Thanks!