Computation of analytical invariants of tangent-to-identity diffeomorphisms

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Plan.

1. Introduction.
2. Definition of analytical invariants.
3. Reminders of Borel summability
4. Reminders on resurgence theory.
5. On a generic 1-order difference equation
6. Methods to compute the invariants.

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Context presentation.

Let $\mathcal{T} = \left\{ f \in \mathbb{C}\{x\} \ ; \ f(x) = \lambda x + \mathcal{O}(x^2) \text{ , where } \lambda \in \mathbb{C}^* \right\}$.

Main goal: Describe conjugacy classes of $\mathcal{T}$.

- Hyperbolic case: $|\lambda| \neq 1$.

  $f \in \mathcal{T}$ is holomorphically conjuguated to $x \longmapsto \lambda x$. (Koenig)

- Elliptic case: $|\lambda| = 1$ and $\lambda = e^{2i\pi\theta}$, $\theta \notin \mathbb{Q}$.

  There exists a diophantine condition, by Bryuno, which is optimal and guarantee the linearization of all the f which begin by λ.
  (Siegel, for a non-optimal condition, Bryuno for the optimality and the sufficientness, Rüssman, Yoccoz for the optimality and the necessity, Perez-Marco)

- Parabolic case: $|\lambda| = 1$ and $\lambda = e^{2i\pi\frac{p}{q}}$.

  $f \in \mathcal{T}$ is not conjugated to $x \longmapsto \lambda x$ (excepted if f has a finite order) .

- Near the infinity, any tangent-to-identity difféomorphism is formally conjuguated to :

$$z \longmapsto z + z^{1-p} - \rho z^{1-2p} \text{ , where } p \in \mathbb{N}^* \text{ and } \rho \in \mathbb{C} \text{ .}$$

p and ρ are the two formal invariants of the diffeomorphism.

- We are interrested in form a finer partition than the "formal conjugacy classes", especially in the typical classe $(p\,;\rho) = (1\,;0)$, i.e.

$$f(z) = z + 1 + \mathcal{O}\left(z^{-2}\right) \text{ .}$$

Let us denote $I : z \longmapsto z + 1$.

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Analytical invariants of a diffeormorphism f.

- $$\begin{cases} \exists! f^* \in z + \mathbb{C}\left[\!\left[z^{-1}\right]\!\right] \ , \ f^* \circ f = l \circ f^* \ . \\ \exists! {}^*f \in z + \mathbb{C}\left[\!\left[z^{-1}\right]\!\right] \ , \ f \circ {}^*f = {}^*f \circ l \ . \end{cases}$$
- $\pi_f^+ = f_+^* \circ {}^*f_-$ commutes with l and is invariant by conjugacy:

$$\begin{cases} \pi^+ - id_{\mathbb{C}} \text{ is } 1 - \text{periodic.} \\ \pi^+_{\varphi \circ f \circ \varphi^{-1}} = \pi_f^+ \ . \end{cases}$$

Définition :

The invariants of f, denoted by $(A^+_{2in\pi}(f))_{n \in \mathbb{Z}^*}$, are the Fourier coefficients of $\pi_f^+ - id_{\mathbb{C}}$.

Theorem: (Ecalle, 80')

Two tangent-to-identity diffeomorphisms from the typical conjugacy classe are conjugated if, and only if, they have the same analytical invariants.

Problematic:

1. Numerically compute the $A^+_{2in\pi}(f)$'s for all $n \in \mathbb{Z}^*$.
2. Have a better combinatoric understanding of these invariants.

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The transformation $\mathcal{B}$, called "formal Borel transform" is defined on $z^{-1}\,\mathbb{C}\left[\!\left[z^{-1}\right]\!\right]$ and valued in $\mathbb{C}[\![\zeta]\!]$:

$$\begin{array}{rccc} \mathcal{B}: & z^{-1}\,\mathbb{C}\left[\!\left[z^{-1}\right]\!\right] & \longrightarrow & \mathbb{C}[\![\zeta]\!] \\ & \displaystyle\sum_{n\geq 0}\frac{c_n}{z^{n+1}} & \longmapsto & \displaystyle\sum_{n\geq 0}\frac{c_n}{n!}\zeta^n \, . \end{array}$$

Notation: $\widehat{\varphi} = \mathcal{B}(\widetilde{\varphi})$.

Properties:

Let $(\widetilde{\varphi};\widetilde{\psi}) \in z^{-1}\,\mathbb{C}\left[\!\left[z^{-1}\right]\!\right]$.

Then:

1. $\mathcal{B}(\partial_z\widetilde{\varphi})(\zeta) = -\zeta\widehat{\varphi}(\zeta)$.
2. $\mathcal{B}(\widetilde{\varphi}\circ I)(\zeta) = e^{-\zeta}\widehat{\varphi}(\zeta)$, where $I:\mathbb{C}\longrightarrow\mathbb{C}$, $I(z) = z+1$.
3. $\mathcal{B}(\widetilde{\varphi}\cdot\widetilde{\psi})(\zeta) = \left(\widehat{\varphi}\star\widehat{\psi}\right)(\zeta)$.

Définition

For any $\theta \in \mathbb{R}$, we say that $\widetilde{\varphi} \in z^{-1}\mathbb{C}\left[\!\left[z^{-1}\right]\!\right]$ is Borel-summable in the direction θ, and we denote $\widetilde{\varphi} \in \mathcal{S}_{\mathcal{B},\theta}$, when the following conditions are satisfied:

1. $\widehat{\varphi} = \mathcal{B}(\widetilde{\varphi})$ can be analytically extended on a neighborhood Ω of $e^{i\theta}\mathbb{R}_+$.
2. there exist two positive constants C and τ such that for all $\zeta \in \Omega$,

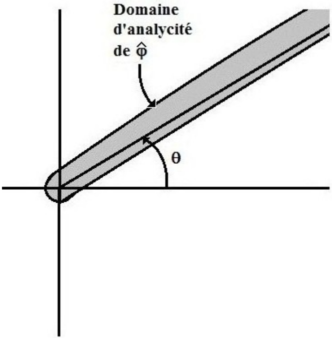
$$|\widehat{\varphi}(\zeta)| \leq Ce^{\tau|\zeta|} \ . \tag{1}$$

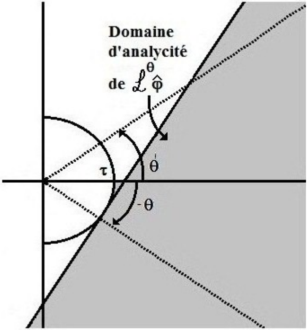
In this case, the Borel sum, denoted by $S^{\theta}(\widetilde{\varphi})$, is defined by:

$$S^{\theta}(\widetilde{\varphi})(z) = \mathcal{L}^{\theta}\left(\mathcal{B}(\widetilde{\varphi})\right)(z) = \int_0^{e^{i\theta}\infty} \mathcal{B}(\widetilde{\varphi})(\zeta)e^{-z\zeta}\, d\zeta \ . \ .$$

Such a sum is automatically an analytical function on the half-plane $\{z \in \mathbb{C} \ ; \ \Re e\left(e^{i\theta}z\right) > \tau\}$.

Holomorphic domain of $S^\theta(\widetilde{\varphi})$.





Borel summation (Continuation) .

Examples:

1. If $\widetilde{\varphi}(z) = \sum_{n\geq 0} \frac{(-1)^n}{z^{n+1}}$, then $S^0(\widetilde{\varphi})(z) = \frac{1}{z+1}$.

   Thus, $1 - 1 + 1 - 1 + \cdots = \frac{1}{2}$.
2. $\widetilde{\varphi}(z) = \sum_{n\geq 0} \frac{n+1}{z^{n+1}}$ is not Borel summable, but $\widetilde{\varphi}(z) = \sum_{n\geq 0} \frac{(-1)^n(n+1)}{z^{n+1}}$ is:

   $$1 - 2 + 3 - 4 + \cdots = \frac{1}{4} .$$

   From this, Ramanujan deduced: $1 + 2 + 3 + 4 + \cdots = -\frac{1}{12}$.

This process is satisfactory, in the following sense:

1. If $\widetilde{\varphi} \in \mathbb{C}\left\{z^{-1}\right\}$ is well-defined on a neighborhood Ω of infinity, then, the Borel sums $S^\theta(\widetilde{\varphi})$ coincide with $\widetilde{\varphi}$ on Ω, for all directions $\theta \in \mathbb{R}$.
2. $S^\theta : \mathcal{S}_{\mathcal{B},\theta} \longrightarrow \bigcup_{\tau>0} \mathcal{H}\left(\mathcal{P}^\theta(\tau)\right)$ is an injective homomorphism which commutes with the derivation.
3. If $\widetilde{\varphi} \in \mathcal{S}_{\mathcal{B},\theta}$, then $\widetilde{\varphi}$ is the asymptotic expansion, near infinity, of $S^\theta(\widetilde{\varphi})$.

Sum up by a diagram.

Formal model:

$$\widetilde{\varphi}(z) = \sum_{n=0}^{+\infty} \frac{c_n}{z^{n+1}} \in z^{-1}\mathbb{C}[\![z^{-1}]\!] \cap \mathcal{S}_{\mathcal{B},\theta} .$$

(formal power series, Borel summable in the direction θ) .

$$\xrightarrow{\mathcal{B}}$$

Convolutive model:

$$\widehat{\varphi}(\zeta) = \sum_{n=0}^{+\infty} \frac{c_n}{n!}\zeta^n .$$

(function that can be analytically exten- -ded on a neighborhood of $e^{i\theta}\mathbb{R}_+$ and that has at most exponential growth) .

$$\xrightarrow{\mathcal{L}^\theta}$$

Geometric model:

$$S^\theta(\widetilde{\varphi})(z) = \mathcal{L}^\theta\widehat{\varphi}(z) = \int_0^{e^{i\theta}\infty} \widehat{\varphi}(\zeta)e^{-\zeta z}d\zeta .$$

(analytical function defined on the half-plane $\{z \in \mathbb{C} \; ; \; \Re e\,(e^{i\theta}z) > 0\})$.

$$\xrightarrow{\text{asymptotic expansion near infinity.}}$$ (back to the Formal model)

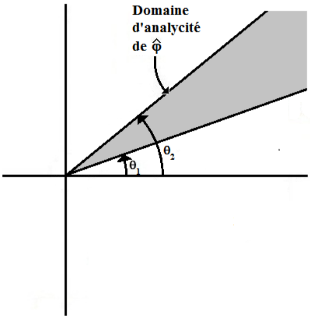
Sectorial resummation

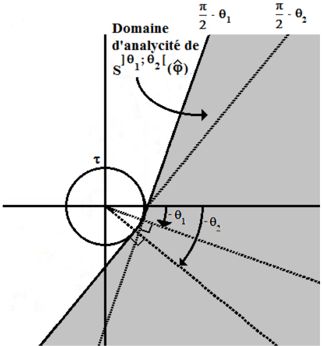
Definition

A formal power series $\widetilde{\varphi}$ is said to be uniformly Borel-summable in the interval of direction $]\theta_1;\theta_2[$ when the following conditions are satisfied:

1. $\widehat{\varphi}$ can be analytically extended to a neighborhood of $\Omega = \{z \in \mathbb{C}\ ;\ \theta_1 < arg\ z < \theta_2\}$.
2. there exist two positive constants C and τ such that for all $\zeta \in \Omega$,

$$|\widehat{\varphi}(\zeta)| \leq Ce^{\tau|\zeta|} \ . \tag{2}$$





Why do we need to work in the Borel plan?

1. In the mutiplicative plan, we have to do some Fourier analysis.

   In the Borel plan de Borel, the invariants are localised in a point.

2. The Borel plan allows us to deal with groups containing $\mathcal{T}$ (*e.g.* this of formal series whose Gevrey growth is 2^-): even if they don't have an interpretation in the multiplicative plan, they have some non trivial conjugacy classes and some invariants which are similar to tangent-to-identity diffeomorphisms (Ecalle, 90').

3. Using the Borel plan will gives us some numerical methods to compute with high precision the analytical invariants.

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- **Simple singularities.**

Let $\widehat{\varphi}$, defined and holomorphic on an open set D.

$\widehat{\varphi}$ has a simple singularity at $\omega \in \mathbb{C}$ adherent to D when $\widehat{\varphi}$ can be expanded near ω as

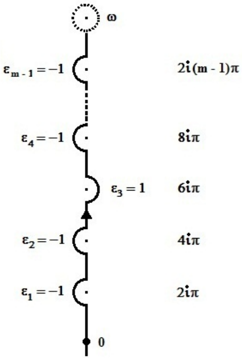
$$\widehat{\varphi}(\zeta) \underset{(\zeta \longrightarrow \omega)}{=} \frac{C}{2i\pi(\zeta-\omega)} + \frac{1}{2i\pi}\widehat{\Phi}(\zeta-\omega)\log(\zeta-\omega) + reg(\zeta-\omega)\ . \quad (3)$$

We will write this egality in a simple way:

$$\mathrm{sing}_{\omega}\widehat{\varphi} = C\delta + \widehat{\Phi} \in \mathbb{C}\delta \oplus \mathbb{C}\{\zeta\}\ . \quad (4)$$

- **Endlessly continuable germs.**

A holomorphic germ φ at the origin is an endlessly continuable germ when, for any finite broken line L, there exists a finite set $\Omega_L \subset L$ of singularities such that φ has an analytic continuation along all the possible paths obtained by following L and getting around each point of Ω_L turning left or right.



- **Simple resurgent functions.**

  In the convolutive model, S-resurgent functions are defined to be endlessly continuable holomorphic germs at the origin, with simple singularities.

  In the formal model, S-resurgent functions are formal power series in $z^{-1}\mathbb{C}[[z^{-1}]]$ such that their Borel transform is an S-resurgent function in the convolutive model.

<u>**Notation:**</u> RES $=$ set of S-Simple resurgent functions.

<u>**Examples:**</u>
For any tangent-to-identity diffeomorphism f, f^* est ${}^*f$ are S-resurgent functions: their Borel transform are uniformly defined on $\widetilde{\mathbb{C} - 2\pi i\mathbb{Z}}$.

Reminders on Alien derivations

For all $\omega \in 2i\pi\mathbb{Z}^*$, we define a derivation, relatively to the convolution product which "measures" the singularities near ω :

$$\Delta_\omega : \mathrm{RES} \longrightarrow \mathrm{RES} \ ,$$

$$\forall \omega \in 2i\pi\mathbb{Z}^* \ , \ \Delta_\omega(\widehat{\varphi}) = \sum_{\varepsilon=(\varepsilon_{\pm 1};\cdots;\varepsilon_{\pm(m-1)})\in\{+1;-1\}^{|m|-1}} \frac{p(\varepsilon)!\, q(\varepsilon)!}{m!} \mathrm{sing}_\omega(\mathrm{cont}_{\gamma(\varepsilon)}\widehat{\varphi}) \ .$$

Definition

In a simpler way, we define $\Delta_\omega^+ : \mathrm{RES} \longrightarrow \mathrm{RES}$ by only considering some right distortion: it is not a derivation.

<u>**Reminders:**</u> The invariants of f, denoted by $(A^+_{2in\pi}(f))_{n\in\mathbb{Z}^*}$, are the Fourier coefficients of $\pi_f^+ - id_{\mathbb{C}}$.

Property

$$\forall \omega \in 2i\pi\mathbb{Z}^* \ , \ \Delta_\omega^+({}^*f) = A_\omega^+(f)\ \partial_z{}^*f + \mathcal{O}(\partial_z^2){}^*f \ .$$

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A simple 1-order difference equation

- Let $f(z) = z + 1 + a(z)$, $a \in z^{-2}\mathbb{C}\left\{z^{-1}\right\}$.

  Fact: Find ${}^*f(z) = z + \varphi(z) \in z + \mathbb{C}\{z^{-1}\}$ such that $f \circ {}^*f = {}^*f \circ I$

$$\Updownarrow$$

  Solve $\varphi(z+1) - \varphi(z) = a\big(z + \varphi(z)\big) = a(z) + \varphi(z)a'(z) + \cdots$.

- **Goal of the section:**

  Study the difference equation (in $\widetilde{\varphi}$) :

$$\widetilde{\varphi}(z+1) - \widetilde{\varphi}(z) = a(z) \ ,$$

  where $a \in z^{-2}\mathbb{C}\left\{z^{-1}\right\}$ is a fixed germ of holomorphic function.

  1. Solving this equation leads us to S-resurgent functions, which turn out to be particularly simple.
  2. The method used here, when iterated in a suitable way, will be usefull to compute the analytical invariants.

Using the Borel transform, with the notation $\widehat{\varphi} = \mathcal{B}(\widetilde{\varphi})$:

1. $e^{-\zeta}\widehat{\varphi}(\zeta) - \widehat{\varphi}(\zeta) = \widehat{a}(\zeta)$, therefore $\widehat{\varphi}(\zeta) = \dfrac{\widehat{a}(\zeta)}{e^{-\zeta} - 1}$.

2. $\widehat{\varphi}$ is a meromorphic function on $\mathbb{C}$ with poles located on $2i\pi\mathbb{Z}^*$: $\widehat{a}$ is a entire function vanishing at 0 and of exponential type at the origine in all directions.

Therefore, the equation $\widetilde{\varphi}(z+1) - \widetilde{\varphi}(z) = a(z)$ has a unique solution in $z^{-1}\mathbb{C}[\![z^{-2}]\!]$:

$$\widetilde{\varphi} = \mathcal{B}^{-1}\left(\frac{\widehat{a}(\zeta)}{e^{-\zeta} - 1}\right) .$$

This is consequently a S-résurgent function satisfying:
$\Delta_\omega \widehat{\varphi} = -\widehat{a}(\omega) = (1 - e^{-\omega})\widehat{\varphi}(\omega)$, for all $\omega \in 2i\pi\mathbb{Z}^*$.

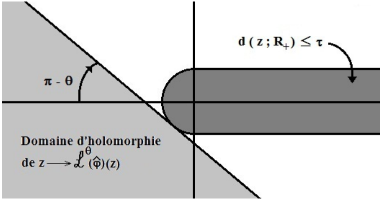
Sectori l resummation of the solutions.

$\widetilde{\varphi}$ is Borel summable in directions $\theta \in \left]-\frac{\pi}{2};\frac{\pi}{2}\right[\cup \left]\frac{\pi}{2};\frac{3\pi}{2}\right[$.

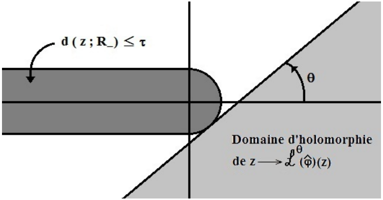
The principe of sectorial resummation gives two analytic solutions, φ^+ et φ^-, defined by:

$$\varphi^+ = \mathcal{L}^\theta\left(\frac{\widehat{a}(\zeta)}{e^{-\zeta}-1}\right),\ \theta \in \left]-\frac{\pi}{2};\frac{\pi}{2}\right[\quad \text{et} \quad \varphi^- = \mathcal{L}^\theta\left(\frac{\widehat{a}(\zeta)}{e^{-\zeta}-1}\right),\ \theta \in \left]\frac{\pi}{2};\frac{3\pi}{2}\right[.$$

defined on $\begin{cases} \mathcal{D}^+ = \mathbb{C} - \{z \in \mathbb{C}\ ;\ d(z\,;\mathbb{R}_-) \leq \tau\}\ . \\ \mathcal{D}^- = \mathbb{C} - \{z \in \mathbb{C}\ ;\ d(z\,;\mathbb{R}_+) \leq \tau\}\ . \end{cases}$



Holomorphic domain of
the sectorial resummation φ^+ .

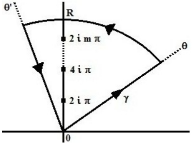


Holomorphic domain of
the sectorial resummation φ^- .

Let us compute the difference $\varphi^+ - \varphi^-$ in each connected component of $\mathcal{D}^+ \cap \mathcal{D}^-$.

Residus calculus gives us:

$$\begin{aligned}\int_\gamma \widehat{\varphi}(\zeta) e^{-z\zeta}\, d\zeta &= \sum_{k=1}^{m} 2i\pi \mathrm{Res}\left(z \longmapsto \frac{\widehat{a}(\zeta) e^{-\omega z}}{e^{-\zeta} - 1} \; ; \; 2ik\pi \right) \\ &= -\sum_{k=1}^{m} 2i\pi \widehat{a}(2ik\pi) e^{-2ik\pi z} \; .\end{aligned}$$



Therefore: $\forall z \in \mathbb{C} \; , \; \Im m\, z < -\tau \; , \; \varphi^+(z) - \varphi^-(z) = - \sum_{\omega \in 2i\pi\mathbb{N}^*} 2i\pi \widehat{a}(\omega) e^{-\omega z} \; .$

Similarly, $\forall z \in \mathbb{C} \; , \; \Im m\, z > \tau \; , \; \varphi^+(z) - \varphi^-(z) = - \sum_{\omega \in -2i\pi\mathbb{N}^*} 2i\pi \widehat{a}(\omega) e^{-\omega z} \; .$

We can notice that $\begin{cases} \varphi^+(z) = -\displaystyle\sum_{k=0}^{+\infty} a(z+k) \; . \\ \\ \varphi^-(z) = \displaystyle\sum_{k=1}^{+\infty} a(z-k) \; . \end{cases}$

So, if $a(z) = \dfrac{1}{z^p}$ for $p \geq 2$, we find out an Eisenstein series and is Fourier expansion:

$$\forall z \in \mathcal{D}^+ \cap \mathcal{D}^-, \quad \mathcal{T}e^p(z) \;=\; \sum_{k \in \mathbb{Z}} \frac{1}{(z+k)^p} \;=\; \varphi^-(z) - \varphi^+(z)$$

$$= \begin{cases} -\dfrac{(-2i\pi)^p}{(p-1)!} \displaystyle\sum_{k=1}^{+\infty} k^{p-1} e^{2ik\pi z} & , \text{ si } \Im m\, z > \tau \; . \\ \\ \dfrac{(2i\pi)^p}{(p-1)!} \displaystyle\sum_{k=1}^{+\infty} k^{p-1} e^{-2ik\pi z} & , \text{ si } \Im m\, z < -\tau \; . \end{cases}$$

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Context reminders.

We are interrested in tangent-to-identity diffeomorphisms:

$$f(z) = z + 1 + \mathcal{O}\left(\frac{1}{z^2}\right) .$$

We have defined $l : z \longmapsto z + 1$.

- $$\begin{cases} \exists! f^* \in z + \mathbb{C}\left[\!\left[z^{-1}\right]\!\right] \ , \ f^* \circ f = l \circ f^* \ . \\ \exists! {}^*f \in z + \mathbb{C}\left[\!\left[z^{-1}\right]\!\right] \ , \ f \circ {}^*f = {}^*f \circ l \ . \end{cases}$$
- f^* et ${}^*f$ are S-resurgent functions.
- $\pi_f^+ = f_+^* \circ {}^*f_-$ commuts with l and is invariant by conjugacy.
- Fourier coefficients of $\pi^+ - id_{\mathbb{C}}$ are the invariants $A_{2in\pi}^+(f)$.

Problematic:

1. Numerically compute the $A_{2in\pi}^+(f)$'s for all $n \in \mathbb{Z}^*$.
2. Have a better combinatoric understanding of these invariants.

- **Method 0 :** Fourier Analysis in the multiplicativ plane.

$$A_{2in\pi}^{+}(f) = \lim_{p \longrightarrow +\infty} \int_{[z\,;z+1]} \left(l^{\circ(-p)} \circ f^{\circ(2p)} \circ l^{\circ(-p)}(z) - z \right) e^{-2in\pi z}\, dz\ .$$

- **Method 1 :** Coefficient Asymptotics.
- **Method 2 :** Resurgent analysis in the Borel plan and conformal transform.
- **Method 3 :** Universal method

Method 3 : Notations.

- Let $\mathbb{N}_k = \{n \in \mathbb{N}\,; n \geq k\}$, pour $k \in \mathbb{N}$.
- $E^\star$ is the set of finite sequences whose elements are in E .
  The empty sequence is denoted by $\emptyset$.
- Let f by of the following type:

$$f(z) = z + 1 + \sum_{n \geq 3} \frac{a_n}{z^{n-1}} \ .$$

$A_\bullet$ is defined on $\mathbb{N}_3^\star$ by : $A_{\underline{s}} = a_{s_1} \cdots a_{s_r}$.
$\mathcal{A}_\bullet$ is defined on $(\mathbb{N}_3^\star - \{\emptyset\})^\star$ by: $\mathcal{A}_{\underline{\boldsymbol{s}}} = A_{\underline{s}^1} \cdots A_{\underline{s}^r}$.

We denote $\mathcal{S}_b^\star = \{\underline{s} \in {\mathbb{N}_1}^\star \; ; \; s_1 \geq 2\}$.

$$\mathcal{S}_{be}^\star = \{\underline{s} \in {\mathbb{N}_1}^\star \; ; \; s_1 \geq 2 \text{ and } s_r \geq 2\} .$$

Multizetas definition

The multiple zeta values are complex numbers defined by

$$\mathcal{Z}e^{\underline{s}} = \sum_{0<n_r<\cdots<n_1<+\infty} \frac{1}{{n_1}^{s_1}\cdots {n_r}^{s_r}}$$

where $\underline{s} \in \mathcal{S}_b^\star$.

Multitangent definition

The multitangent functions are defined by

$$\mathcal{T}e^{\underline{s}}(z) = \sum_{-\infty<n_r<\cdots<n_1<+\infty} \frac{1}{(n_1+z)^{s_1}\cdots(n_r+z)^{s_r}}$$

where $\underline{s} \in \mathcal{S}_{be}^\star$ and $z \in \mathbb{C} - \mathbb{Z}$.

Notations: $\mathcal{M}ZV = Vect_{\mathbb{Q}}(\mathcal{Z}e^{\underline{s}})_{\underline{s}\in\mathcal{S}_d^\star}$, $\mathcal{M}TGF = Vect_{\mathbb{Q}}(\mathcal{T}e^{\underline{s}})_{\underline{s}\in\mathcal{S}_{df}^\star}$.

Method 3 : universal method.

We expand each $A^+_{2in\pi}(f)$ as an entire function of f, i.e. of the Taylor coefficients of f:

Theorem (O. Bouillot, 11)

Let f be a convergent tangent-to-identity diffeomorphism defined by

$$f(z) = z + 1 + \sum_{n \geq 3} \frac{a_n}{z^{n-1}} \; .$$

Then :

1. There exists explicit coefficients $\tau^\bullet$, defined on $(\mathbb{N}_3^\star - \{\emptyset\})^\star$, valued in the algebra $\mathcal{M}TGF$ s.t. :

$$\pi_f^+ = \sum_{\underline{s} \in \left(\mathbb{N}_3^\star - \{\emptyset\}\right)^\star)} \tau^{\underline{s}} \, \mathcal{A}_{\underline{s}} \; .$$

2. If $n \in \mathbb{Z}^*$, there exists explicit coefficients $\widehat{\tau}_n^\bullet$, defined on $(\mathbb{N}_3^\star - \{\emptyset\})^\star$, valued in the algebra $\mathcal{M}ZV$ for all , s.t. :

$$\forall n \in \mathbb{Z}^*, \; A_n^+(f) = \sum_{\underline{s} \in \left(\mathbb{N}_3^\star - \{\emptyset\}\right)^\star)} \widehat{\tau}_n^{\underline{s}} \, \mathcal{A}_{\underline{s}} \; .$$

Method 3: Sketch of proof

- We can write $\pi_f^+(z) - z = \sum_{\underline{\boldsymbol{n}} \in \mathbb{Z}^\star} U^{\underline{\boldsymbol{n}}}\, \Gamma_{\underline{\boldsymbol{n}}} \cdot z$, where :

$$U^{\underline{\boldsymbol{n}}} = \begin{cases} 1 & \text{, if } n_r < n_{r-1} < \cdots < n_1 \ . \\ 0 & \text{, otherwise.} \end{cases}$$

$$\begin{cases} \Gamma_n \cdot \varphi(z) = \sum_{k \geq 1} \dfrac{(\varphi(z+n) - z)^k}{k!} \partial_z^k \varphi(z) \\ \Gamma_{\underline{\boldsymbol{n}}} = \Gamma_{n_r} \circ \cdots \circ \Gamma_{n_1} \end{cases}$$

- We compute $\Gamma_{\underline{\boldsymbol{n}}} \cdot z$.

- We compute the Fourier coefficients of $\pi_f^+ - id$, using the normal convergence of $\sum_{\underline{\boldsymbol{n}} \in \mathrm{seq}(\mathbb{Z})} U^{\underline{\boldsymbol{n}}}\, \Gamma_{\underline{\boldsymbol{n}}} \cdot z$, on every compact subset of a upper or lower half-plane.

- We conclude with the previous expression of $\pi_f^+ - id$ and $A_n^+(f)$.

$\square$

Remark: A monotangent function is a multitangent function with length 1 .

Let $m(\underline{\boldsymbol{s}}) = \max(s_1 ; \cdots ; s_r)$, for all $\underline{\boldsymbol{s}} \in \mathbb{N}_1^\star$.

Property: Reduction of Convergent Multitangent Functions into Monotangent Functions.

$$\forall \underline{\boldsymbol{s}} \in \mathcal{S}^\star \ , \ \exists (z_1 ; \cdots ; z_{m(\underline{\boldsymbol{s}})}) \in \mathcal{MZV}^{m(\underline{\boldsymbol{s}})} \quad , \ \mathcal{T}e^{\underline{\boldsymbol{s}}} = \sum_{k=1}^{m(\underline{\boldsymbol{s}})} z_k \mathcal{T}e^k \ .$$

Sketch of proof:

1. Partial fraction expansion of $\dfrac{1}{(n_1 + X)^{s_1} \cdots (n_r + X)^{s_r}}$.

Method 3: Reduction into Monotangent Functions.

Remark: A monotangent function is a multitangent function with length 1 .

Let $m(\underline{s}) = \max(s_1 ; \cdots ; s_r)$, for all $\underline{s} \in \mathbb{N}_1^\star$.

Property: Reduction of Convergent Multitangent Functions into Monotangent Functions.

$$\forall \underline{s} \in \mathcal{S}^\star \ , \ \exists (z_2 ; \cdots ; z_{m(\underline{s})}) \in \mathcal{M}ZV^{m(\underline{s})-1} \ , \ \mathcal{T}e^{\underline{s}} = \sum_{\substack{\cancel{k=1} \\ k=2}}^{m(\underline{s})} z_k \mathcal{T}e^k \ .$$

Sketch of proof:

1. Partial fraction expansion of $\dfrac{1}{(n_1 + X)^{s_1} \cdots (n_r + X)^{s_r}}$.
2. Using an analytic argument:

$$\forall z \in \mathbb{C} - \mathbb{R} \ , \ |\mathcal{T}e^{\underline{s}}(z)| \leq 4r \left(\frac{2}{|\Im m\, z|} \right)^{s_1 + \cdots + s_r - r - 1} \frac{e^{-\pi |\Im m\, z|}}{1 - e^{-\pi |\Im m\, z|}} \ .$$

$\square$

Weight 4

$$\mathcal{T}e^{2,2} = 2\zeta(2)\mathcal{T}e^{2} \ .$$

Weight 5

$$\mathcal{T}e^{2,3} = -3\,\zeta(3)\mathcal{T}e^{2} + \zeta(2)\mathcal{T}e^{3} \ .$$

$$\mathcal{T}e^{3,2} = 3\,\zeta(3)\mathcal{T}e^{2} + \zeta(2)\mathcal{T}e^{3} \ .$$

$$\mathcal{T}e^{2,1,2} = 0 \ .$$

Weight 6

$$\mathcal{T}e^{2,4} = \frac{8}{5}\zeta(2)^2\mathcal{T}e^{2} - 2\zeta(3)\mathcal{T}e^{3} + \zeta(2)\mathcal{T}e^{4} \ .$$

$$\mathcal{T}e^{3,3} = -\frac{12}{5}\zeta(2)^2\mathcal{T}e^{2} \ .$$

$$\mathcal{T}e^{4,2} = \frac{8}{5}\zeta(2)^2\mathcal{T}e^{2} + 2\zeta(3)\mathcal{T}e^{3} + \zeta(2)\mathcal{T}e^{4} \ .$$

$$\mathcal{T}e^{2,2,2} = \frac{8}{5}\zeta(2)^2\mathcal{T}e^{2} \ .$$

$$\mathcal{T}e^{2,1,3} = -\frac{2}{5}\zeta(2)^2\mathcal{T}e^{2} + \zeta(3)\mathcal{T}e^{3} \ .$$

$$\mathcal{T}e^{3,1,2} = -\frac{2}{5}\zeta(2)^2\mathcal{T}e^{2} - \zeta(3)\mathcal{T}e^{3} \ .$$

$$\mathcal{T}e^{2,1,1,2} = \frac{4}{5}\zeta(2)^2\mathcal{T}e^{2} \ .$$

- Choice of a 1-parameter family of tangent-to-identity diffeomorphism:

$$f(z) = l \circ g(z) = z + 1 + \frac{\alpha}{z^2} + \mathcal{O}(z^{-3}) \ , \tag{5}$$

where :

$$\begin{aligned} l(z) &= z + 1 \ . \\ g(z) &= \left(\exp(\alpha z^{-2} \partial_z) \right) \cdot z = z \left(1 + 3\,\alpha z^{-3}\right)^{1/3} \ . \end{aligned}$$

- Let define $\sigma : z \mapsto -z$.
We see that $\sigma \circ f \circ \sigma$ is conjuguated to f^{-1}.
So, $\pi_f^+ - id_{\mathbb{C}}$ is an even function, *i.e.* π_f^+ will be expressed by the even monotangent functions:

$$\mathcal{T}e^{s_1}(z) = \sum_{n \in \mathbb{Z}} \frac{1}{(z+n)^{s_1}} \ , \qquad \text{with } s_1 \in 2\mathbb{N}^* \ .$$

- π_f^+ expressed by the multitangent functions :

$$\begin{aligned}\pi_f^+ - id_{\mathbb{C}} = &+\alpha \cdot \mathcal{T}a^{2}\\ &-\alpha^2 \cdot \left(2\mathcal{T}a^{3,2}\right)\\ &+\alpha^3 \cdot \left(4\mathcal{T}a^{3,3,2} + 6\mathcal{T}a^{4,2,2}\right)\\ &-\alpha^4 \cdot \left(8\mathcal{T}a^{3,3,3,2} + 12\mathcal{T}a^{3,4,2,2} + 12\mathcal{T}a^{4,2,3,2} + 24\mathcal{T}a^{4,3,2,2} + 24\mathcal{T}a^{5,2,2,2}\right)\\ &+\mathcal{O}(\alpha^5)\end{aligned}$$

- π_f^+ en fonction des monotangentes et des multizêtas non réduits :

$$
\begin{aligned}
\pi_f^+ - id_{\mathbb{C}} = \quad & +\alpha \cdot \mathcal{T}e^2 \qquad\qquad + \alpha^2 \mathcal{T}e^2 \cdot \big(-6\,\zeta(3)\big) \\
& +\alpha^3 \cdot \mathcal{T}e^2 \cdot \big(+26\,\zeta(6) + 2\,\zeta(4,2) - 16\,\zeta(2,4) + 36\,\zeta(3,3)\big) \\
& +\alpha^3 \cdot \mathcal{T}e^4 \cdot \big(+\zeta(4) - 2\,\zeta(2,2)\big) + \alpha^3 \cdot \mathcal{T}e^6 \cdot \left(-\frac{1}{3}\,\zeta(2)\right) \\
& +\alpha^4 \cdot \mathcal{T}e^2 \cdot \big(-86\,\zeta(9) + 52\,\zeta(5,4) + 68\,\zeta(7,2) - 34\,\zeta(4,5) \\
& \qquad\qquad - 156\,\zeta(3,6) + 224\,\zeta(2,7) - 246\,\zeta(6,3) - 16\,\zeta(4,3,2) \\
& \qquad\qquad + 56\,\zeta(5,2,2) - 216\,\zeta(3,3,3) + 96\,\zeta(3,2,4) - 12\,\zeta(4,2,3) \\
& \qquad\qquad + 64\,\zeta(2,3,4) - 12\,\zeta(3,4,2) + 64\,\zeta(2,5,2) + 96\,\zeta(2,4,3)\big) \\
& +\alpha^4 \cdot \mathcal{T}e^4 \cdot \big(+2\,\zeta(7) + 34\,\zeta(2,5) + 30\,\zeta(5,2) - 10\,\zeta(3,4) - 18\,\zeta(4,3) \\
& \qquad\qquad + 12\,\zeta(2,2,3) + 16\,\zeta(2,3,2) + 20\,\zeta(3,2,2)\big) \\
& +\alpha^4 \cdot \mathcal{T}e^6 \cdot \left(+\frac{13}{3}\,\zeta(5) + 4\,\zeta(2,3) + \frac{14}{3}\,\zeta(3,2)\right) \\
& +\mathcal{O}(\alpha^5)
\end{aligned}
$$

Method 3: Examples of invariant computations (continuation)

- π_f^+ en fonction des monotangentes et des multizêtas réduits :

$$
\begin{aligned}
\pi_f^+ - id_{\mathbb{C}} = \quad & +\alpha \cdot \mathcal{T}e^2 \cdot \qquad\qquad +\alpha^2 \cdot \mathcal{T}e^2 \cdot \left(-6\,\zeta(3)\right) \\
& +\alpha^3 \cdot \mathcal{T}e^2 \cdot \left(-\frac{32}{5}\zeta(2)^3 + 36\,\zeta(3)^2 \right) + \alpha^3 \cdot \mathcal{T}e^4 \cdot \left(-\frac{1}{5}\zeta(2)^2 \right) \\
& +\alpha^3 \cdot \mathcal{T}e^6 \cdot \left(-\frac{1}{3}\zeta(2)^2 \right) \\
& +\alpha^4 \cdot \mathcal{T}e^2 \cdot \left(\frac{576}{5}\zeta(3)\zeta(2)^3 - 216\,\zeta(3)^3 - 210\,\zeta(9) \right) \\
& +\alpha^4 \cdot \mathcal{T}e^4 \cdot \left(14\,\zeta(7) + \frac{18}{5}\zeta(3)\zeta(2)^2 \right) \\
& +\alpha^4 \cdot \mathcal{T}e^6 \cdot \left(6\,\zeta(2)\zeta(3) - \frac{10}{3}\zeta(5) \right) \\
& +\mathcal{O}(\alpha^5)
\end{aligned}
$$

- The invariants $A_{\pm 2\pi i}(f)$ seen as an entire functions of the parameter α:

$$A_{\pm 2\pi i}(f) = \sum_{n \in \mathbb{N}^*} c_n \alpha^n \text{ is an entire function of exponential type in } \alpha^{1/2}.$$

Its first coefficients are:

$$\begin{aligned}
c_1 &= -\mathbf{39}.4784176043574344753\ldots \\
c_2 &= +\mathbf{284}.7318264428106410205\ldots \\
c_3 &= -\mathbf{788}.4456763395103611766\ldots \\
c_4 &= +\mathbf{1183}.6708974792155533 10\ldots \\
c_5 &= -\mathbf{1124}.0131018827372145 16\ldots \\
c_6 &= +\mathbf{738}.5776097731620314 53\ldots \\
c_7 &= -\mathbf{356}.3887910169968 09\ldots \\
c_8 &= +\mathbf{131}.76870562724\ldots \\
c_9 &= -\mathbf{38}.5440209553\ldots \\
c_{10} &= +\mathbf{9}.1457604\ldots \\
c_{11} &= -\mathbf{1}.796\ldots \\
c_{12} &= +\mathbf{0}.3\ldots
\end{aligned}$$

Conclusion

- The analytical invariants and the horn map have some explicit formula.
- All the analytical invariants can be numerically computed with high precision, in an efficient way.
- The analytical invariant suggest some interesting arithmetical questions on the multitangents and on the multizetas, as well as on the relations between the multitangents and the multizetas

Conclusion

- The analytical invariants and the horn map have some explicit formula.
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Thank you for your attention!