Modulus of analytic classification of a generic germ of analytic family unfolding a parabolic point of codimension k

Christiane Rousseau, ETDS 35 (2015)

Work done in part with C. Christopher, P. Mardešić, R. Roussarie and L. Teyssier

Structure of the lecture

- Statement of the problem
- The preparation of the family
- Construction of a modulus of analytic classification
- The theorem of analytic classification

Statement of the problem

We consider germs of generic k-parameter families f_ϵ of diffeomorphisms unfolding a parabolic point of codimension k

$$f_0(z) = z + z^{k+1} + o(z^{k+1})$$

When are two such germs conjugate?

Conjugacy of two germs of families

Two germs of families of diffeomorphisms f_ϵ and $\tilde{f}_{\tilde{\epsilon}}$ are conjugate it there exists $r, \rho > 0$ and analytic functions

$$h : \mathbb{D}_\rho \to \mathbb{C}, \qquad H : \mathbb{D}_r \times \mathbb{D}_\rho \to \mathbb{C}$$

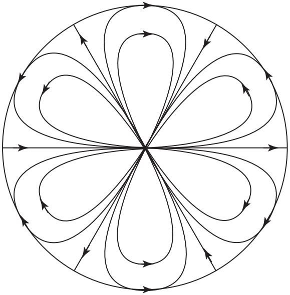
such that

- h is a diffeomorphism and for each fixed ϵ, $H_\epsilon = H(\cdot, \epsilon)$ is a diffeomorphism;
- for all $\epsilon \in \mathbb{D}_\rho$ and for all $z \in \mathbb{D}_r$, then

$$\tilde{f}_{h(\epsilon)} = H_\epsilon \circ f_\epsilon \circ (H_\epsilon)^{-1}$$

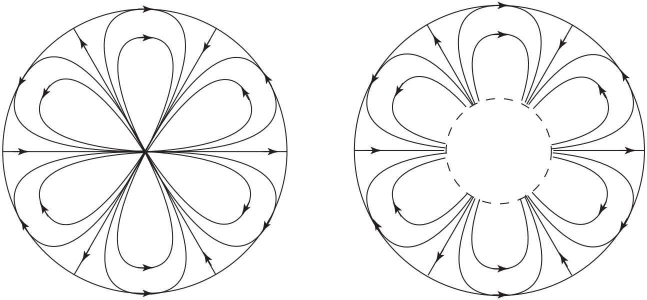
The choice of $\mathbb{D}_r$

$\mathbb{D}_r$ is chosen so that the behaviour of f_0 on the boundary is as in



The choice of $\mathbb{D}_\rho$ in parameter space

$\mathbb{D}_\rho$ is chosen sufficiently small so that f_ϵ has the same behaviour near the boundary as f_0. In particular, all fixed points of f_ϵ remain inside the disk.



A natural strategy: the use of normal forms

A germ of generic k-parameter family f_ϵ unfolding a parabolic point of codimension k is formally conjugate to the time-1 map of a vector field

$$v_\epsilon = \frac{P_\epsilon(z)}{1+a(\epsilon)z^k}\frac{\partial}{\partial z}$$

where

$$P_\epsilon(z) = z^{k+1} + \epsilon_{k-1}z^{k-1} + \cdots + \epsilon_1 z + \epsilon_0$$

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Problem: the change to normal form diverges. What does it mean?

Can we exploit the formal normal form despite its divergence?

Let us look at the case $k = 1$:

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Two singular points $\pm\sqrt{\epsilon}$ with eigenvalues

$$\mu_\pm = \frac{\pm 2\sqrt{\epsilon}}{1 \pm a(\epsilon)\sqrt{\epsilon}}$$

The parameter is an analytic invariant of the vector field!

Indeed, we have

$$\frac{1}{\mu_+} + \frac{1}{\mu_-} = a(\epsilon)$$

$$\frac{1}{\mu_+} - \frac{1}{\mu_-} = \frac{1}{\sqrt{\epsilon}}$$

Hence, can we hope to bring the system to a "prenormal" form in which the parameter is invariant?

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Advantage: a conjugacy between prepared families must preserve the *canonical* parameters.

Theorem

We consider a diffeomorphism with a parabolic point of codimension k:

$$f_0(z) = z + z^{k+1} + o(z^{k+1})$$

For any generic k-parameter unfolding f_η, there exists an analytic change of coordinate and parameter $(z,\eta) \mapsto (Z,\epsilon)$ in a neighborhood of the origin transforming the family into the prepared form

$$F_\epsilon(Z) = Z + P_\epsilon(Z)(1 + Q_\epsilon(Z) + P_\epsilon(Z)K(Z,\epsilon))$$

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such that, if $Z_1, \dots Z_{k+1}$ are the fixed points, then

$$F'_\epsilon(Z_j) = \exp\left(\frac{P'_\epsilon(Z_j)}{1 + a(\epsilon)Z_j^k}\right)$$

This determines almost uniquely the parameters!

The only freedom will be inherited from a rotation of order k in Z

$$Z \mapsto \tau Z; \qquad \tau^k = 1$$

which yields the corresponding change on ϵ:

$$(\epsilon_{k-1}, \epsilon_{k-2} \dots, \epsilon_0) \mapsto (\tau^{2-k}\epsilon_{k-1}, \tau^{1-k}\epsilon_{k-2}, \dots, \tau\epsilon_0)$$

The strategy

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For each $\epsilon \in \mathbb{D}_\rho$ with ρ chosen sufficiently small we compare f_ϵ with the "model" given by the formal normal form, namely the time-1 map of

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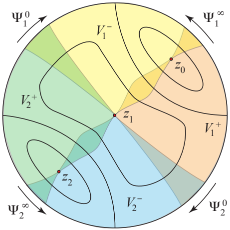
Comparing is constructing a change of coordinates to normal form.

A global analytic comparison on $\mathbb{D}_r$ does not exist

Hence, we cover $\mathbb{D}_r$ with $2k$ sectors $V^{\pm}_{j,\epsilon}$. Over each sector the normalizing change of coordinate is almost unique (up to a symmetry of the model, which is a time t map of v_ϵ.)



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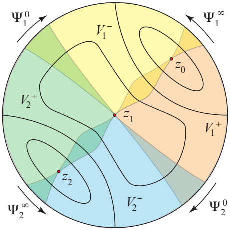

The modulus is given by the comparison of the normalizations over the intersections of the sectors. It is a *symmetry* of the model.

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We first describe the modulus over each sectoral domain W_s. It is simply an unfolding of the Ecalle-modulus.

The construction of the sectors in x-space

This construction is governed by the geometry of the polynomial vector field

$$w_\epsilon = iP_\epsilon(z)\frac{\partial}{\partial z}$$

where

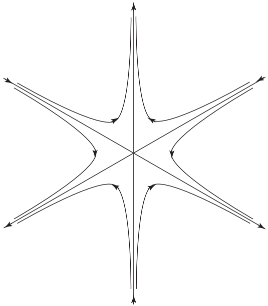
$$P_\epsilon(z) = z^{k+1} + \epsilon_{k-1}z^{k-1} + \cdots + \epsilon_1 z + \epsilon_0$$

which has been studied by Douady and Sentenac.

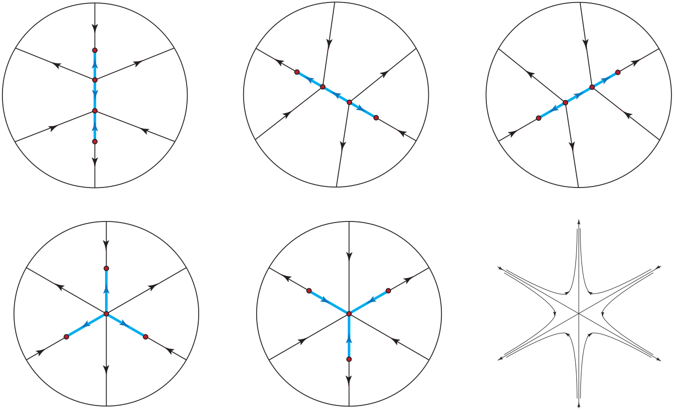
The generic strata of $w_\epsilon = iP_\epsilon(z)$

Generically (except on a set of real codimension 1) the separatrices of ∞ land at singular points.

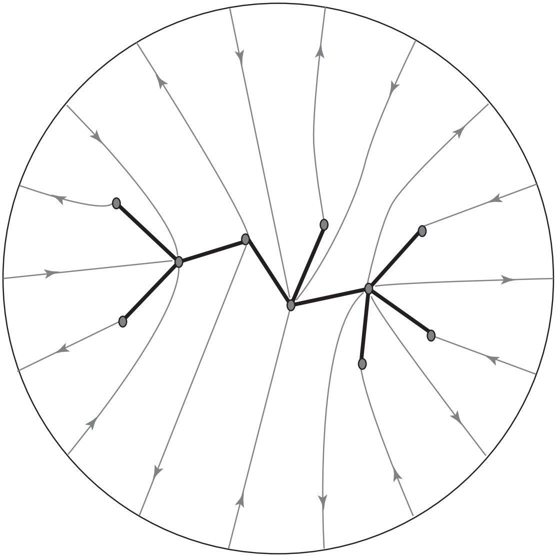
There are $C_k = \frac{\binom{2k}{k}}{k+1}$ topological ways to achieve this.



Example: the five cases when $k = 3$



Another example



Changing coordinate

We change to the coordinate Z, which is the time coordinate of the vector field $w_\epsilon(z) = iP_\epsilon(z)\frac{\partial}{\partial z}$

$$Z = \begin{cases} -\frac{i}{kz^k}, & \epsilon = 0, \\ \sum_{j=1}^{k+1} \frac{\log(z-z_j)}{P'_\epsilon(z)}, & z_j \text{ distinct.} \end{cases}$$

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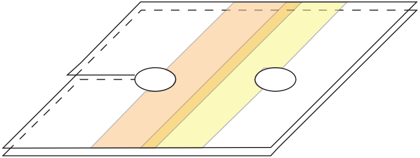
Note that Z is multivalued on a k-sheeted Riemann surface for $\epsilon \neq 0$

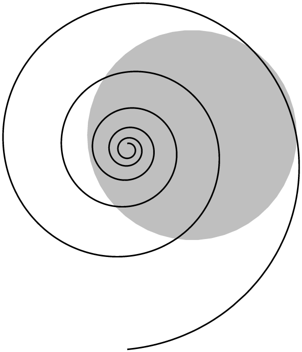
This allows defining the sectors

The $2k$ sectors in z-space correspond to vertical strips Z-space.
We enlarge them so that they overlap.



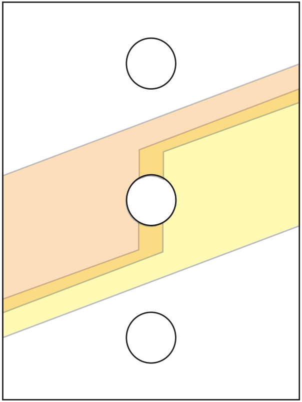
But we cannot cover all parameter values this way

Indeed, we study a local problem... The separatrices may cut the disk $\mathbb{D}_r$ before landing at a singular point...

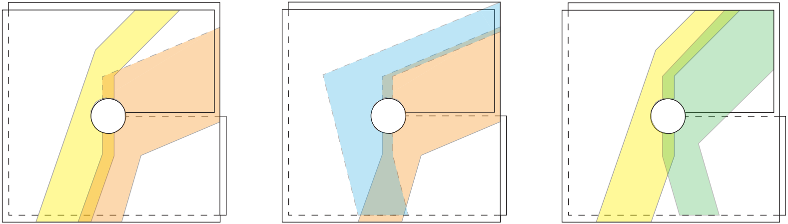


Covering all parameter values

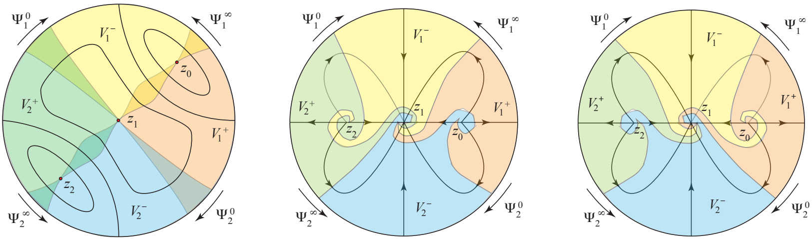
Then, we bend the strips near the infinite ends (in different sheets)



We could use different slopes at the two ends of a strip



In this way we cover all Σ_0



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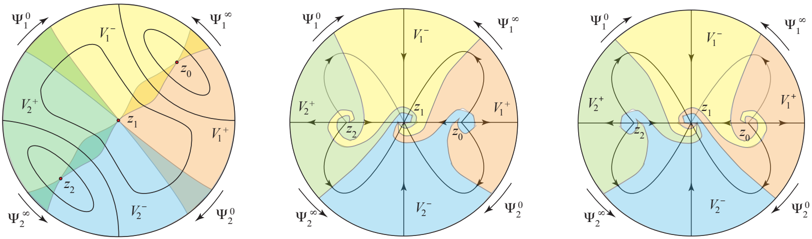

The parameter values in the intersections of sectoral domains are covered by non-equivalent configurations of sectors.

Constructing the Fatou coordinates

We lift the map $f_\epsilon(z)$ to $F_\epsilon(Z)$. We have

$$\begin{cases} |F_\epsilon(Z) - Z - 1| < Kr \\ |F'_\epsilon(Z) - 1| < Kr^{k+1} \end{cases}$$

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$$\begin{cases} |F_\epsilon(Z) - Z - 1| < Kr \\ |F_\epsilon'(Z) - 1| < Kr^{k+1} \end{cases}$$

This allows constructing changes of coordinates $\Phi^{\pm}_{j,\epsilon}$ on the strips conjugating $F_\epsilon(Z)$ to $Z \mapsto Z+1$.

These are unique up to left composition with translations. They depend on the sectoral domain W_s.

Dependence of the Fatou coordinates on ϵ

On a sectoral domain, the Fatou coordinates can be chosen to depend analytically on ϵ with fixed limit at $\epsilon = 0$.

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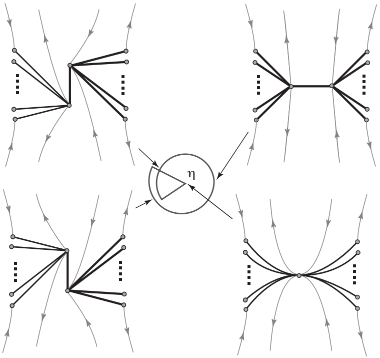
Within a sectoral domain W_s, the Fatou coordinates can be chosen to have a continuous limit at a generic point of $\Delta(\epsilon) = 0$.



The modulus of analytic classification over a sectoral domain W_s

It is given by comparing the Fatou coordinates on the intersections of the strips.

$$\begin{cases} \Psi^{\infty}_{j,\epsilon} = \Phi^{-}_{j,\epsilon} \circ (\Phi^{+}_{j,\epsilon})^{-1} \\ \Psi^{0}_{j,\epsilon} = \Phi^{-}_{j+1,\epsilon} \circ (\Phi^{+}_{j,\epsilon})^{-1} \end{cases}$$

which are defined respectively on half-planes $\mathrm{Re}(Z) > Y_0$ or $\mathrm{Re}(Z) < -Y_0$.

Normalized Fatou coordinates

The maps $\Psi_{j,\epsilon,s}^{0,\infty}$ commute with T_1 and hence, can be expanded in Fourier series. It is possible to choose the Fatou coordinates so that the constant terms of the Fourier series be

$$A_{j,\epsilon}^0 = -A_{j,\epsilon}^{\infty} = \pi i a(\epsilon)/k.$$

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$$A^0_{j,\epsilon} = -A^\infty_{j,\epsilon} = \pi i a(\epsilon)/k.$$

Then, there remains one degree of freedom in the choice of the Fatou coordinates: for instance, the image of a given point $Z_0 = p_\epsilon(z_0)$ for $z_0 \in V^+_{1,\epsilon,s}$.

The modulus of analytic classification over a sectoral domain W_s

Let us consider two prepared families $f_\epsilon(z)$ and $\tilde{f}_{\tilde{\epsilon}}(z)$.

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$$\tilde{\epsilon}_j \mapsto \epsilon_j = \exp(2\pi i(j-1)m/k)\tilde{\epsilon}_j$$

we can suppose that $\epsilon = \tilde{\epsilon}$.

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we can suppose that $\epsilon = \tilde{\epsilon}$.

If the $f_\epsilon(z)$ and $\tilde{f}_\epsilon(z)$ have unfolded moduli $(\Psi^\infty_{j,\epsilon,s}, \Psi^0_{j,\epsilon,s})_{\epsilon\in W_s}$ and $(\widetilde{\Psi}^\infty_{j,\epsilon,s}, \widetilde{\Psi}^0_{j,\epsilon,s})_{\epsilon\in W_s}$ over W_s, with

$$\Psi^{0,\infty}_{j,\epsilon,s} = T_{-C_s(\epsilon)} \circ \widetilde{\Psi}^{0,\infty}_{j,\tilde{\epsilon},s} \circ T_{C_s(\epsilon)},$$

then they are conjugate over W_s.

Proof

Under the condition

$$\Psi_{j,\epsilon,s}^{0,\infty} = T_{-C_s(\epsilon)} \circ \widetilde{\Psi}_{j,\tilde{\epsilon},s}^{0,\infty} \circ T_{C_s(\epsilon)},$$

the conjugacy is given by

$$H_{\epsilon,s} = p_\epsilon \circ \left[(\widetilde{\Phi}_{j,\epsilon,s}^{\pm})^{-1} \circ T_{C_s(\epsilon)} \circ \Phi_{j,\epsilon,s}^{\pm} \right] \circ p_\epsilon^{-1}.$$

This map is uniform in z.

The theorem of analytic classification

Theorem

Let $f_\epsilon(z)$ and $\tilde{f}_{\tilde{\epsilon}}(z)$ be two generic prepared families, each unfolding a germ of parabolic diffeormorphism of codimension k. Then, they are analytically conjugate if and only if they have the same modulus, i.e.

- *for each j, $\epsilon_j = \exp(2\pi i(j-1)m/k)\tilde{\epsilon}_j$;*
- $a(\epsilon) = \tilde{a}(\tilde{\epsilon})$;
- *for each sectoral domain W_s, there exist constants $C_s(\epsilon)$ such that*

$$\Psi^{0,\infty}_{j,\epsilon,s} = T_{-C_s(\epsilon)} \circ \widetilde{\Psi}^{0,\infty}_{j+m,\tilde{\epsilon},s} \circ T_{C_s(\epsilon)},$$

with the limit $C_s(0)$ independent of s.

The proof

The direct part is straightforward.

For the converse, we suppose that the two families have the same modulus.

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For the converse, we suppose that the two families have the same modulus.

Three steps:

1. treating the generic values of $\epsilon \in \Sigma_0$ for which all singular points are distinct (the discriminant of $P_\epsilon(z)$ is nonzero;
2. extending to values of ϵ for which two singular points coallesce;
3. using Hartogs' theorem for the remaining values.

First step

We must glue the conjugacies $H_{\epsilon,s}$ defined on the different sectoral domains in a global H_ϵ.

$$H_{\epsilon,s}^{-1} \circ H_{\epsilon,s'}$$

is a symmetry of f_ϵ. Such a symmetry is of the form $f_\epsilon^{\alpha(\epsilon)}$.

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Generically, the set of symmetries (i.e. the set of admissible $\alpha(\epsilon)$) is discrete. This is the case as soon as one $\Psi^0_{\epsilon,j,s}$ or $\Psi^\infty_{\epsilon,j,s}$ is not a translation.

end of first step

1. One of $\Psi^0_{\epsilon,j,s}$ or $\Psi^\infty_{\epsilon,j,s}$ is not a translation. Then, since we can choose $\Psi^{0,\infty}_{0,j,s}$ independent of s, this yields $\alpha(0)=0$, and then $\alpha(\epsilon)=0$. Hence, we get a uniform H_ϵ over Σ_0.

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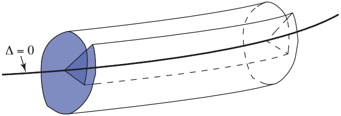
2. All $\Psi^{0,\infty}_{\epsilon,j,s}$ are translations. We can normalize all Fatou coordinates by asking for instance that $\Phi^+_{1,\epsilon,s}(Z_0) = Z_0$ for a fixed base point Z_0. If $z_0 = p_\epsilon(Z_0)$, then

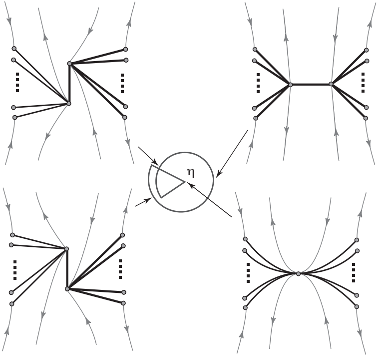
$$H_{\epsilon,s}(z_0) = z_0,$$

from which it follows that H_ϵ is independent of s.

Second step

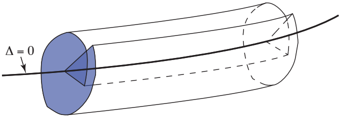
We must extend H_ϵ to the generic points of $\Delta(\epsilon) = 0$. This is done by approching these points through one sectoral domain W_s.

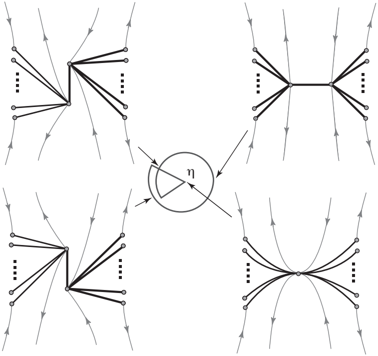




end of second step

As before we can argue that $H^{-1}_{\epsilon e^{2\pi i}} \circ H_\epsilon$ is a symmetry of f_ϵ, which has a bounded limit at the generic points of $\Delta(\epsilon) = 0$.





Third step

We extend H_ϵ at the remaining points by Hartogs' theorem.

The *parametric resurgence phenomenon*: an example

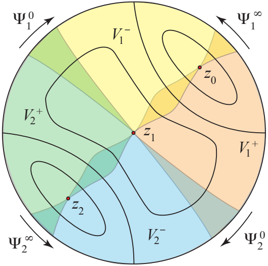
We consider sequences of parameter values for which the renormalized return map at a fixed point has a fixed resonant multiplier and fixed Lavaurs maps. The renormalized return map is

$$L_{12} \circ \psi^0_{2,\epsilon} \circ L_{10} \circ \psi^0_{1,\epsilon}.$$

For small ϵ, the fixed point is nonlinearizable as soon as

$$L_{12} \circ \psi^0_{2,0} \circ L_{10} \circ \psi^0_{1,0}$$

is nonlinearizable at the origin.



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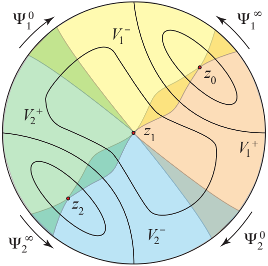

Hence, the nonlinearizability can be read from the modulus at $\epsilon = 0$.

The end

The proof

We consider a diffeomorphism with a parabolic point of codimension k:

$$f_0(z) = z + z^{k+1} + o(z^{k+1})$$

A k-parameter unfolding can be written in the form

$$f_\eta(z) = z + p_\eta(z) g_\eta(z),$$

with $g_\eta(z) = 1 + O(\eta, z)$.

Using the Weierstrass division theorem on the rest and a translation in z allows to write f_η in the form

$$f_\eta(z) = z + p_\eta(z)\big(1 + q_\eta(z) + p_\eta(z)h_\eta(z)\big)$$

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with

$$p_\eta(z) = z^{k+1} + \nu_{k-1}(\eta)z^{k-1} + \nu_1(\eta)z + \nu_0(\eta)$$

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with

$$p_\eta(z) = z^{k+1} + \nu_{k-1}(\eta)z^{k-1} + \nu_1(\eta)z + \nu_0(\eta)$$

and

$$q_\eta(z) = c_0(\eta) + c_1(\eta)z + \cdots + c_k(\eta)z^k.$$

Genericity condition

The Jacobian

$$\frac{\partial \nu}{\partial \eta}$$

is invertible.

Since

$$f_\eta(z) = z + p_\eta(z)\left(1 + q_\eta(z) + p_\eta(z)h_\eta(z)\right)$$

the fixed points z_j of f_η are the zeroes of p_η.

The strategy

The formal normal form is the time one map of a vector field

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Hence the the fixed points of f_η must be sent to the singular points Z_j of V_ϵ.
Moreover we need have

$$f_\eta'(z_j) = \exp(V_\epsilon'(Z_j))$$

How do we find the formal invariant $a(\epsilon)$?

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There exists a polynomial $r_\eta(z)$ of degree $\leq k$ such that at the points z_j we have

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(Such a polynomial is found by the Lagrange interpolation formula for distinct z_j. The limit exists when two fixed points coallesce (codimension 1 case). We can fill in for the other values of η by Hartogs's Theorem.)

The reparameterization

By Kostov theorem, there exists a change of coordinate and parameter transforming the vector field:

$$p_\eta(z)(1+r_\eta(z))\frac{\partial}{\partial z}=v_\eta(z)$$

into:

$$P_\epsilon(Z)/(1+a(\epsilon)Z^k)\frac{\partial}{\partial Z}=V_\epsilon(Z),$$

where

$$P_\epsilon(Z)=Z^{k+1}+\epsilon_{k_1}Z^{k-1}+\epsilon_1 Z+\epsilon_0.$$

We apply this change of coordinate and parameter to f_η.

Claim: this brings f_η to a prepared form F_ϵ

- It sends the zeros z_j of $p_\eta(z)$ to the zeroes of $P_\epsilon(Z)$. Since the z_j are the fixed points of f_η, their images are the fixed points Z_j of F_ϵ.

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- Hence

$$\begin{aligned} F_\epsilon(Z) &= Z + P_\epsilon(Z)K_\epsilon(Z) \\ &= Z + P_\epsilon(Z)(1 + Q_\epsilon(Z) + P_\epsilon(Z)H_\epsilon(Z)) \end{aligned}$$

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- Let be a fixed point. Then

$$F'_\epsilon(Z_j) = \lambda_j = f'_\eta(z_j) = \exp(v'_\eta(z_j)) = \exp(V'_\epsilon(Z_j))$$

which is what we need for a prepared family.

The parameters are (almost) canonical

We have

$$
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F_\epsilon(Z) &= Z + P_\epsilon(Z)K_\epsilon(Z) \\
&= Z + P_\epsilon(Z)(1 + Q_\epsilon(Z) + P_\epsilon(Z)H_\epsilon(Z))
\end{aligned}
$$

Claim: P_ϵ, Q_ϵ and ϵ are unique up to the change

$$Z \mapsto \tau Z; \qquad \tau^k = 1$$

and the corresponding change on ϵ:

$$(Z, \epsilon_{k-1}, \epsilon_{k-2} \dots, \epsilon_0) \mapsto (\tau Z, \tau^{2-k}\epsilon_{k-1}, \tau^{1-k}\epsilon_{k-2}, \dots, \tau\epsilon_0)$$

The proof

Let us suppose that two prepared families $f_\epsilon(z)$ and $\tilde{f}_{\tilde{\epsilon}}(\tilde{z})$ are conjugate under a map $(\tilde{\epsilon},\tilde{z}) = (h(\epsilon), H_\epsilon(z))$:

$$\tilde{f}_{h(\epsilon)} = H_\epsilon \circ f_\epsilon \circ H_\epsilon^{-1}$$

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f_ϵ

Fixed points z_j are those of $v_\epsilon(z) = P_\epsilon(z)/(1+az^k)\frac{\partial}{\partial z}$

$\tilde{f}_{\tilde{\epsilon}}$

Fixed points $\tilde{z}_j$ are those of $\tilde{v}_{\tilde{\epsilon}}(\tilde{z}) = \tilde{P}_{\tilde{\epsilon}}(\tilde{z})/(1+a\tilde{z}^k)\frac{\partial}{\partial \tilde{z}}$

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Note that the formal invariants are the same.

Then H_ϵ sends the fixed points z_j to the fixed points $\tilde{z}_j$. Hence

$$H_\epsilon^*(\tilde{v}_{h(\epsilon)})(z) = P_\epsilon(z)U_\epsilon(z)\frac{\partial}{\partial z} = w_\epsilon(z)$$

where $U \neq 0$.

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v_ϵ and w_ϵ have the same singular points with same eigenvalues! Hence

$$w_\epsilon = P_\epsilon(z)\left(\frac{1}{1+az^k} + P_\epsilon(z)M_\epsilon(z)\right)\frac{\partial}{\partial z}$$
$$= v_\epsilon(1+P_\epsilon(z)N_\epsilon(z))\frac{\partial}{\partial z}.$$

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Then $(K_\epsilon^{-1}\circ H_\epsilon)^*(\tilde{v}_{h(\epsilon)}) = v_\epsilon$. The result follows from the following theorem proved with L. Teyssier.

Theorem (RT)

We consider a germ of an analytic change of coordinates $\Psi : (z,\epsilon) = (z,\epsilon_0,\dots,\epsilon_{k-1}) \mapsto (\varphi_\epsilon(z), h_0(\epsilon),\dots,h_{k-1}(\epsilon)) = (z,h)$ *at* $(0,0,\cdots,0) \in \mathbb{C}^{1+k}$*. The following assertions are equivalent :*

1. *the families* $\left(\frac{P_\epsilon(z)}{1+a(\epsilon)z^k}\frac{\partial}{\partial z}\right)_\epsilon$ *and* $\left(\frac{P_h(z)}{1+\tilde{a}(h)z^k}\frac{\partial}{\partial z}\right)_h$ *are conjugate under* Ψ*,*
2. *there exist* τ *with* $\tau^k = 1$ *and* $T(\epsilon)$ *an analytic germ such that, if* $R_\tau(z) = \tau z$
   - $\varphi_\epsilon(z) = \Phi_{v_\epsilon}^{T(\epsilon)} \circ R_\tau(z)$
   - $\epsilon_j = \tau^{j-1} h_j(\epsilon)$,
   - $a(\epsilon) = \tilde{a}(h(\epsilon))$.

Reduction to the case $\tau = 1$

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So we can compose $\Psi(z, \epsilon)$ with R_τ and the corresponding change of parameters $\epsilon_j = \tau^{j-1} h_j(\epsilon)$ and only discuss the composed family.

Hence we can suppose that $\Psi(z, \epsilon)$ is such that $\varphi_0'(0) = 1$.

The case $\epsilon = 0$

It is easy to check that the only changes of coordinates tangent to the identity which preserve v_0 are the maps $\Phi_{v_0}^t$.

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Indeed, such changes of coordinates have the form $z(1+m_t(z^k))$ with $m_t(z) = tz^k + o(z^k)$. The function $m_t(z)$ is completely determined by $m_t'(0) = t$. This is exactly the form of the family $\Phi_{v_0}^t$.

Reduction to the case $\frac{\partial^{k+1}\varphi_\epsilon}{\partial z^{k+1}}(0) = 0$

We correct φ to

$$G(z,t,\epsilon) := \Phi^t_{v_\epsilon} \circ \varphi_\epsilon(z)$$

with $t(\epsilon)$ well chosen.

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Let

$$H(z,t,\epsilon) := \frac{\partial^{k+1}G}{\partial z^{k+1}}(z,t,\epsilon)$$

$$K(t,\epsilon) := H(0,t,\epsilon)$$

K is analytic. There exists t_0 such that $K(t_0,0) = 0$. Also,

$$\frac{\partial K}{\partial t}(t_0,0) = (k+1)! \neq 0.$$

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Composing φ_ϵ with $\Phi^{t(\epsilon)}_{X_\epsilon}$ we can suppose that the original family Ψ is such that $\frac{\partial^{k+1}\varphi_\epsilon}{\partial z^{k+1}}(0) = 0$.

The rest of the argument is an infinite descent

We introduce the ideal

$$I = \langle \epsilon_0, \dots, \epsilon_{k-1} \rangle.$$

We have

$$\varphi_\epsilon(z) := z + \sum_{n \geq 0} f_n(\epsilon) z^n$$

where $f_n \in I$ and $f_{k+1} \equiv 0$.
We must solve

$$\left(1 + a(\epsilon) z^k\right)\left(\varphi_\epsilon^{k+1}(z) + h_{k-1}\varphi_\epsilon^{k-1}(z) + \dots + h_0\right) \\ - \left(1 + \tilde{a}(h)\varphi_\epsilon^k(z)\right)\left(z^{k+1} + \epsilon_{k-1}z^{k-1} + \dots + \epsilon_0\right)\varphi_\epsilon'(z) = 0.$$

It is then clear that $h_j(\epsilon) \in I$ and $f_j(\epsilon) \in I$.

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It is then clear that $h_j(\epsilon) \in I$ and $f_j(\epsilon) \in I$.

Let $g_j z^j$ be the term of degree j. We will play with the infinite set of equations $g_j = 0$, $j \geq 0$.

The equations $g_j = 0$ with $0 \leq j \leq k-1$ yield

$$h_j - \epsilon_j \in I^2,$$

since all other terms in the expression of g_j belong to I^2.

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The equation $g_{k+j} = 0$ with $0 \le j \le k$ yields $f_j \in I^2$.

Looking at the linear terms in the equations $g_\ell = 0$ with $\ell > 2k+1$ yields $f_{\ell-k} \in I^2$.
So we have that $f_j \in I^2$ for all j.

The general step by induction

We suppose that $h_j - \epsilon_j \in I^n$ when $0 \leq j \leq k-1$ and $f_j \in I^n$ whenever $j \geq 0$.

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To show that $h_j - \epsilon_j \in I^{n+1}$ for $0 \le j \le k-1$ we consider again the corresponding equations $g_j = 0$, where the only linear terms are $h_j - \epsilon_j$. Hence all other terms of the equation belong to I^{n+1} yielding $h_j - \epsilon_j \in I^{n+1}$.

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For the same reason the equation $g_{k+j} = 0$ with $0 \leq j \leq k$ yields $f_j \in I^{n+1}$ and the equations $g_\ell = 0$ with $\ell > 2k+1$ yields $f_{\ell-k} \in I^{n+1}$.