Framework, results and open problems

for Log Sobolev Inequalities in noncommutative spaces

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Non-commutative L_p spaces

- $\{\mathcal{A}, ||\cdot||, \{\mathcal{A}_\Lambda, \Lambda \subset\subset \mathbb{Z}^d\}\}$ the inductive limit C^* algebra over a finite dimensional complex matrix algebra $\mathbf{M}$, with a norm $||\cdot||$.

Positive elements	$\mathcal{A}^+ \equiv \{f \in \mathcal{A}: \exists g \in \mathcal{A} \quad f = g^* g\}$.
Selfadjoint elements	$\mathcal{A}_{\mathrm{sa}} \equiv \{f \in \mathcal{A}: f = f^*\}$.
Local elements	$\mathcal{A}_o \equiv \{f \in \cup \mathcal{A}_\Lambda, \Lambda \subset\subset \mathbb{Z}^d\}$.

- **$\mathbf{Tr}_X: \mathcal{A} \longrightarrow \mathcal{A}_{X^\complement}$ partial traces**

$$\mathbf{Tr}_X f = \int \nu_X (dU)\, U^* f U$$

where $\nu_X (dU)$ - the normalised Haar's measure on the group of all unitaries U in $\mathcal{A}_X$.

Partial trace preserves unit and positivity.

- **A state**

$\omega(f) \equiv \mathbf{Tr}\,(\rho f) \quad , \qquad \rho \in \mathcal{A}_o^+ \equiv \mathcal{A}_o \cup \mathcal{A}^+, \ \ \mathbf{Tr}\,(\rho) = 1$.

- **$\mathbb{L}_2(\omega, s)$ scalar products**

$$< f\,,\, g >_{\mathbb{L}_2(\omega,s)} \equiv \mathbf{Tr}(\rho^{1-s} f^* \rho^s g) = \omega\Big(f^* \alpha_{\frac{\mathrm{is}}{2}}(g)\Big)\,.$$

where $\alpha_t(g) \equiv \rho^{-\mathrm{it}} g \rho^{\mathrm{it}}$.

REM:

(i) $< f\,,\, g >_{\mathbb{L}_2(\omega,s=0)} = \omega(f^* g)$

(ii) $< f\,,\, g >_{\mathbb{L}_2(\omega,s=1)} = \omega(g f^*)$

(iii) $< f\,,\, g >_{\mathbb{L}_2(\omega,s=1/2)} \equiv \mathbf{Tr}\Big(\rho^{1/2} f^* \rho^{1/2} g\Big) = \mathbf{Tr}\Big(\Big(\rho^{1/4} f^* \rho^{1/4}\Big)\Big(\rho^{1/4} g \rho^{1/4}\Big)\Big)$

REM: *Bogolubov-Kubo-Mori scalar product*

$$< f\,,\, g >_{\mathrm{BKM}} \equiv \int_0^1 ds\, \mathbf{Tr}\Big(\rho^{\frac{1-s}{2}} f^* \rho^{\frac{s}{2}} g\Big)$$

- **$\mathbb{L}_p(\omega, s)$ norms** [Zo'82−85]...

$$\|f\|^p_{\mathbb{L}_p(\omega,s)} := \mathbf{Tr}\left|\rho^{\frac{1-s}{p}} f \rho^{\frac{s}{p}}\right|^p, \qquad p \in [1, \infty)$$

for $f \in \mathcal{A}_o$.

Some Properties

- Duality and Ho"lder Inequality $p, q \in (1, \infty), \frac{1}{p} + \frac{1}{q} = 1, \quad s \in [0, 1]$

  $|< f\,, g >_s| \leq \| f \|_{p,s} \cdot \| f \|_{q,s}\,,$

- Interpolation Property, both in $p \in [1, \infty]$ & $s \in [0, 1]$.

- **Flow through Banach spaces and Entropy**

  Embedings of Positive Cones : For $p, q \in (1, \infty)$, define $I_{p,q} : \mathbb{L}_q^+(\omega) \to \mathbb{L}_p^+(\omega)$, $(s = \frac{1}{2})$

$$I_{p,q}(f) \equiv \rho^{-\frac{1}{2p}} \left(\rho^{\frac{1}{2q}} f \rho^{\frac{1}{2q}} \right)^{q/p} \rho^{-\frac{1}{2p}}$$

  One has

$$\| I_{p,q}(f) \|_p^p = \| f \|_q^q$$

  Generator of a Flow through Banach spaces

  For $\quad f \in \mathbb{L}_q^+(\omega), \quad \tau \geq 0$

$$\mathbf{T}_q(f) \equiv \frac{d}{d\tau} I_{q+\tau,q}(f)_{|\tau=0} = \rho^{-\frac{1}{2q}} \left(\left(\rho^{\frac{1}{2q}} f \rho^{\frac{1}{2q}} \right) \log \left(\rho^{\frac{1}{2q}} f \rho^{\frac{1}{2q}} \right) \right) \rho^{-\frac{1}{2q}} - \frac{1}{2q} \{ f, \log \rho \}$$

  Covariance Property

  If $\frac{1}{p} + \frac{1}{q} = 1$, $p, q \in (1, \infty)$ and $f \in \mathcal{A}^+$, then

  $\langle I_{p,q}(f), \mathbf{T}_q(f) \rangle_\omega = \frac{2}{q} \langle I_{2,q}(f), \mathbf{T}_2(I_{2,q}(f)) \rangle_\omega$

$$= \mathrm{Tr} \left(\left(\rho^{\frac{1}{2q}} f \rho^{\frac{1}{2q}} \right)^q \left(\log \left(\rho^{\frac{1}{2q}} f \rho^{\frac{1}{2q}} \right) - \frac{1}{q} \log \rho \right) \right)$$

Theorem.

If $f \in \mathcal{A}^{+}$, then $q \mapsto \|f\|_q \equiv \|f\|_{q,s=1/2}$ *is differentiable and*

$$\frac{d}{\mathrm{dq}}\|f\|^q_{q,} = \langle I_{p,q}(f), \mathbf{T}_q(f)\rangle_\omega = \mathrm{Tr}\left(\left(\rho^{\frac{1}{2q}} f \rho^{\frac{1}{2q}}\right)^q \left(\log\left(\rho^{\frac{1}{2q}} f \rho^{\frac{1}{2q}}\right) - \frac{1}{q}\log\rho\right)\right)$$

with $\frac{1}{p} + \frac{1}{q} = 1$, $p, q \in (1, \infty)$.

REM:

Therefore for any continuously differentiable $q = q(t) \in (1, \infty)$, with $\dot{q} = \frac{d}{\mathrm{dt}} q$, *we have*

$$\frac{d}{\mathrm{dq}}\|f\|^q_{q,} = \dot{q}\,\langle I_{p,q}(f), \mathbf{T}_q(f)\rangle_\omega$$

- Convexity[BCL'94]

  $\|f\|_{p,s}^2 \leq (\omega(f))^2 + (p-1)\|f-\omega(f)\|_{p,s}{}^2$

- Infinitesimal Implication[O&Z'99]

  $\|f\|_{p,s}^2 - \|f\|_{2,s}^2 \leq (\|f-\omega(f)\|_{p,s}^2 - \|f-\omega(f)\|_{2,s}^2) + (p-2)\|f-\omega(f)\|_{p,s}^2$

  $$\Rightarrow \qquad (\partial_p \|f\|_{p,s}^2)_{|p=2} \leq (\partial_p \|f-\omega(f)\|_{p,s}^2)_{|p=2} + \|f-\omega(f)\|_{2,s}^2$$

  $$\Rightarrow \qquad \mathrm{Ent}_{\omega,2}(f) \leq \mathrm{Ent}_{\omega,2}(\tilde{f}) + 2\|\tilde{f}\|_{2,\frac{1}{2}}^2$$

  where $\tilde{f} \equiv f - \omega(f)$

  & $\qquad \mathrm{Ent}_{\omega,2}(f) \equiv \mathbf{Tr}\left(\left|\rho^{\frac{1}{4}} f \rho^{\frac{1}{4}}\right|^2 \left(\log\left|\rho^{\frac{1}{4}} f \rho^{\frac{1}{4}}\right|^2 - \log\rho\right)\right)$

- Joint Monotonicity[Al&Z'14] : Set $\|f\|_{\boldsymbol{\eta},p}^p \equiv \|f\|_{\boldsymbol{\eta},p,s}^p \equiv \mathbf{Tr}\left|(\boldsymbol{\eta}^{-1})^{\frac{1-s}{p}} f (\boldsymbol{\eta}^{-1})^{\frac{s}{p}}\right|^p$

  $\|\mathbf{E}(f)\|_{\mathbf{E}(\boldsymbol{\eta}),2n} \leq \|f\|_{\boldsymbol{\eta},2n}$ for any *completely positive map* $\mathbf{E}$.

  [open question for general $p \in (1,\infty)$]

REM: For Gibbs States

$$||\, f\,||^p_{\mathbb{L}_p(\omega_\Lambda,s)} \equiv \mathbf{Tr}\left|\rho_\Lambda^{\frac{1-s}{p}}\, f\, \rho_\Lambda^{\frac{s}{p}}\right|^p,$$

for $f\in\mathcal{A}$ with $\rho_\Lambda\in\mathcal{A}_\Lambda$.

Case $p=2n\in\mathbb{N}$: With $\alpha_t^{(\Lambda)}(g)\equiv\rho_\Lambda^{-\mathrm{it}} g \rho_\Lambda^{\mathrm{it}}$, $f=f^*$ and $s=1/2$

$$||\, f\,||^p_{\mathbb{L}_p(\omega_\Lambda,s)} \equiv \mathbf{Tr}\, \rho_\Lambda\Bigg((\rho_\Lambda^{-1} f \rho_\Lambda)\bigg(\rho_\Lambda^{-\left(1-\frac{1}{2n}\right)} f \rho_\Lambda^{\left(1-\frac{1}{2n}\right)}\bigg)\rho_\Lambda^{-\left(1-\frac{1}{2n}\right)}\, \rho_\Lambda^{\frac{1}{2n}} f \rho_\Lambda^{\frac{1}{2n}} \ldots f \rho_\Lambda^{\frac{1}{2n}}\, f \rho_\Lambda^{\frac{1}{2n}}\Bigg)$$

$$= \mathbf{Tr}\, \rho_\Lambda\Big(\Big(\alpha_{-i}^{(\Lambda)}(f)\Big)\Big(\alpha^{(\Lambda)}_{-i(2n-1)/2n}(f)\Big) \ldots \alpha^{(\Lambda)}_{-i/2n}(f)\Big)$$

$$= \omega_\Lambda\Big(\Big(\alpha_{-i}^{(\Lambda)}(f)\Big)\Big(\alpha^{(\Lambda)}_{-i(2n-1)/2n}(f)\Big) \ldots \alpha^{(\Lambda)}_{-i/2n}(f)\Big)$$

Thus formally as $\omega_\Lambda\to\omega$ and $\alpha^{(\Lambda)}_{t+i\tau}(f)\to\alpha_{t+i\tau}(f)$, $(\tau\in[-\beta,+\beta])$, we can have meaningfull expresion

$$||\, f\,||^p_{\mathbb{L}_p(\omega,s)} \equiv \omega(\,(\alpha_{-i}(f)\,)(\alpha_{-i(2n-1)/2n}(f))\ldots\alpha_{-i/2n}(f)\,).$$

REM : For general case consider

$$||\, f\,||^p_{p,s} \equiv \mathrm{limsup}\, ||\, f\,||^p_{\mathbb{L}_p(\omega_\Lambda,s)}$$

REM: Other Noncommutative Spaces : Noncommutative Orlicz Spaces.

[Al&Z], [St], [Je], [Maj&La]...[$\mathrm{Ku}_{\mathrm{Tr}}$],...

Orlicz functional $\mathbf{\Lambda}_{\Phi}(f)$ - (absolutely) convex, vanishing only at 0,
and

$$\mathbf{\Lambda}_{\Phi}(\lambda f)/\lambda \to_{\lambda\to\infty} \infty \;\&\; \mathbf{\Lambda}_{\Phi}(\lambda f)/\lambda \to_{\lambda\to 0} 0 \text{ for } f \neq 0.$$

E.g.'s

– Δ_2 [Al&Z'07]

$$\mathbf{\Lambda}_{\Phi}(f) = \mathrm{Tr}\,\Phi((\Phi^{-1}(\rho))^{1-s} f (\Phi^{-1}(\rho))^{s})$$

[In large interactivng systems because of Entropic Switch$_{[Z'05]}$]

– $\mathrm{Exp}_{1[Z]}$

$$\mathbf{\Lambda}_{\exp_1}(f) = \sum_{n=0}^{\infty} \frac{1}{n!} \|f\|^n_{\mathbb{L}_n(\omega,s)}$$

– [St], [Je]

$$\mathbf{\Lambda}_{\exp}(f) = \frac{1}{2}\mathrm{Tr}(e^{-H+f} + e^{-H-f} - 2)$$

with $\mathrm{Tr}(e^{-H}) = 1$

[*May be other* L_{2n} functionals $\int_{0<s_1<\ldots<s_{2n}<1} \mathrm{ds}_1 \ldots \mathrm{ds}_{2n}\, Tr(\rho^{1-s_1} f^* \rho^{s_1-s_2} f \ldots f^* \rho^{s_{2n-1}-s_{2n}} f \rho^{s_{2n}})$]

[JaRo'12]

REM: von Neuman Algebras & Noncommutative Integration theory

[Haa'79], [ArM'82], [Terp'81–2], [Hi'81], [Kos'84]...[Ne'74], [Yea'75], ..., [Se'53], [Dix'53]...([PiXu'03])

Markov Semigroups

Semigroup of operators $(P_t)_{t\geq 0}$ (linear or nonlinear)

$P_t:\mathbb{B}\to\mathbb{B}$, where $(\mathbb{B},\|\cdot\|)$ a Banach space ($(\mathcal{A},\|\cdot\|)$, $(\mathbb{L}_p(\omega,s),\|\cdot\|_{p,s})$, Orlicz space, ...);

- $P_t P_s = P_{t+s}$, $t,s\geq 0$;
- $P_{t=0}=\mathrm{id}$;
- $t\mapsto P_t f$ continuous for any $f\in\mathbb{B}$, (strongly, (in op norm, weakly,..., in vNnn algebras)).

Positive : For a proper convex cone $\mathbb{B}^+$

$P_t:\mathbb{B}^+\to\mathbb{B}^+$ ($P_t:\mathcal{A}^+\to\mathcal{A}^+$ & $\pi(P_t\mathcal{A}^+)\subseteq\mathbb{B}^+$?,)

2-Positive

Schwartz Inequailty [Choi'80]

$P_t(f^*f)\geq P_t(f^*)P_t(f)$

n-Positive

$P_t^{(n)}: M_n(\mathcal{A})\to M_n(\mathcal{A})$

$P_t^{(n)}(f\otimes E_{\mathrm{ij}})=P_t(f)\otimes E_{\mathrm{ij}}$

(where $E_{\mathrm{ij}}, i,j=1,...,n$ are matrix units spanning $M_n(\mathbb{C})$), is *positive*.

Completely Positive

$\forall n\in\mathbb{N}$ $P_t^{(n)}: M_n(\mathcal{A})\to M_n(\mathcal{A})$ is *positive*

Unit Preserving

- $P_t\mathbb{1}=\mathbb{1}$, $\forall t\geq 0$;

Symmetric in $\mathbb{L}_2(\omega,s)$ ([SQV'84]+via Dirichlet Forms [AH-K'77], [Ci']+Korean Grp[Pa])

E.g.'s

a) Linear [GoderisMaes'91], [Matsui]$_{\mathrm{GroundStateRepresentation}}$; [BaKoPa'03]$_{\mathrm{Ext}_{\mathrm{classicalIsing}}}$; Using ClassicalGibbsMeasures

b) Gaussian type semigroups ([CiFaLi], [OZa], [Pa...])

c) On ∞-dim algebras [MZ], [MOZ], ...,

d) Diffusion Type (Ho"rmander type Generators) ([LOZ'10])

e) via Drichlet Forms ([Pa'05] avoiding L_1 asymptotic abelianess, ...)

f) No E.g.s of symmetric jump type @ ∞-dim spaces with *non-classical* interaction

g) Nonlinear ([LOZ'13])

$$S_t(f) \equiv e^{-t} f + \int_0^t \mathrm{ds} \log \omega \left(\exp\left(e^{-s} f\right)\right)$$

(nonlinear annealing algorithm to find a ground state)

Markov Generators

Lindblad generators of CP semigroups

$L_{\boldsymbol{X},Q}(f) = \sum_j \left(X_j^* f X_j - \frac{1}{2}\{X_j^* X_j, f\} \right) + i[Q, f]$

In Diffusive form: With $\delta_A(f) \equiv i[A, f] \equiv i(Af - fA)$

Since, for $\lambda, \kappa \in (0, \infty)$

$$-\lambda \delta_{X^*} \delta_X(f) - \kappa \delta_X \delta_{X^*}(f) = \\ (\lambda + \kappa)(X^* f X + X f X^*) + \frac{\lambda + \kappa}{2}(\{X^* X, f\} + \{X X^*, f\}) \\ + \frac{\lambda - \kappa}{2}([X^* X, f] + [X X^*, f])$$

so

$$L_{\boldsymbol{X},Q}(f) + L_{\boldsymbol{X}^*,Q}(f) = \sum_j \left(L_{X_j}(f) + L_{X_j^*}(f) + \delta_{Q_j}(f) \right) \\ = -\sum_j \left(\mu_j \delta_{X_j^*} \delta_{X_j}(f) + \nu_j \delta_{X_j} \delta_{X_j^*}(f) \right) + \sum_j i(\mu_j - \nu_j)(\delta_{X_j^* X_j} + \delta_{X_j X_j^*})\big) + \delta_Q(f)$$

with $\mu_j \in (0, 1)$ and $\nu_j \equiv 1 - \mu_j$.

E.g.s

* Quantum Orstein-Uhlenbeck$_{[CFL'00],[CaSa'08]}$

With $\lambda > \mu > 0$ and $[A_j, A_j^*] = \mathbb{1}$,

$$L_{A_j}(f) \equiv -\frac{1}{2}\mu^2(A_j^*A_j f + fA_j^*A_j - 2A_j^* fA_j) - \frac{1}{2}\lambda^2(A_jA_j^* f + fA_jA_j^* - 2A_j fA_j^*)$$

* Jump Type Generator

$$\mathcal{L}f = \sum_j (E_j(f) - f)$$

with E_j unit preserving CP maps

E.g.

$E_X(f) \equiv \mathrm{Tr}_X(\gamma_X^* f \gamma_X)$

- ($e.g.$ generalised conditional expectations corresponding to classical potentials)

- ([MaZ'95−96] finite range)

Symmetric operators in L_2 [SQV/84 – KMS symmetric ops]

$L_1 - asymptotic\ abelianess$

$\exists \mathcal{A}_0 : \overline{\mathcal{A}_0} = \mathcal{A}\ \forall x, y \in \mathcal{A}_0 \qquad \int_{-\infty}^{+\infty} \mathrm{ds}\, \|[\alpha_s(x), y]\| < \infty$

For h positive definite, integrable, $h(-t + i\beta) = h(t)$

define

$L_{x,h}(y) \equiv \int\ \mathrm{dt}\,\mathrm{ds}\, h(t)(\alpha_s(x)[y, \alpha_{s+t}(x)] + [\alpha_s(x), y]\, \alpha_{s+t}(x))$

Then

$$\Gamma(y) \equiv \frac{1}{2}(L_{x,h}(y^* y) - y^* L_{x,h}(y) - L_{x,h}(y^*) y)$$
$$= \int\ \mathrm{dt}\,\mathrm{ds}\, h(t-s)[\alpha_t(x), y]^* [\alpha_s(x), y] \geq 0$$

and

$$\omega(L_{x,h}(y)^* z) = \omega(z^* L_{x,h}(y))$$

Markov Semigroups on infinite dimensional algebras

Construction and Ergodicity

[MOZ][MZ]$_{\text{SpinSystems}}$,[LOZ]$_{\text{HoermanderType}}$,[MaesG],[Matsui]$_{\text{GroundState}}$

Markovian Quadratic Form for a Markov Generator $\mathcal{L}$

$\Gamma_{\mathcal{L}}(f) \equiv \frac{1}{2}(\mathcal{L}(f^*f) - \mathcal{L}(f^*)f - f^*\mathcal{L}(f))$

REM:

$$P_t(f^*f) - P_t(f^*)P_t(f) = \int_0^t \mathrm{ds}\, \frac{d}{\mathrm{ds}} P_s(P_{t-s}(f^*)P_{t-s}(f)) =$$

$$= \int_0^t \mathrm{ds}\, P_s\, \Gamma_{\mathcal{L}}(P_{t-s}(f))$$

For Lindblad Generator of CP semigroup

$\Gamma_{\mathcal{L}}(f) = \frac{1}{2}\left(|\delta_{X^*}(f)|^2 + |\delta_X(f)|^2\right) \geq 0$

For Jump Type Generator

$$\Gamma_{\mathcal{L}}(f) = \frac{1}{2}\sum_j \left(E_j(f^*f) - f^*E_j(f) - E_j(f^*)f + f^*f\right)$$

$$= \frac{1}{2}\sum_j (E_j(f^*f) - E_j(f^*)E_j(f)) + \frac{1}{2}\sum_j (E_j(f^*)E_j(f) - f^*E_j(f) - E_j(f^*)f + f^*f)$$

$$= \frac{1}{2}\sum_j (E_j(f^*f) - E_j(f^*)E_j(f)) + \frac{1}{2}\sum_j |E_j(f) - f|^2$$

is non-negative, thanks to $E_j(f^*f) - E_j(f^*)E_j(f) \geq 0$ for CP unit preserving maps.

Poincare Inequality : $\exists m \in (0, \infty)\ \forall f \in \mathcal{D}(\mathcal{L})$

$$m < f, f >_{\mathbb{L}_2(\omega,s)} \quad \leq \quad \mathcal{E}_2(f) \qquad\qquad (\mathbf{P})$$

with

$$\mathcal{E}_2(f) \equiv < f, (-\mathcal{L}f) >_{\mathbb{L}_2(\omega,s)}$$

REM:

- U-bounds techniques ...[HZ′10]...

  $$< f, \eta f > \quad \leq \quad \mathcal{E}_2(f) \text{ for some un-bdd } \eta$$

Weak Exponential Bound - 1

$(\mathbf{P}) \Rightarrow$ If "$|||f||| \equiv \sum_j \|\Gamma_j(f)\|$"$< \infty$ & $\omega|f| < \infty$ then $\exists \varepsilon_0 > 0\ \forall \varepsilon < \varepsilon_0$ "$< e^{\varepsilon f} >$"$< \infty$

REM:

In commutative case

$(\mathbf{P}) \Rightarrow$ If $\|f\|_{\mathrm{Lip}} < \infty \int |f| < \infty$, then $\exists \varepsilon_0 > 0\ \forall \varepsilon < \varepsilon_0 \int e^{\varepsilon f} < \infty$

Hypercontractivity in Noncommutative Spaces.

Definition of **Hypercontractivity**:

A Markov semigroup $P_t \equiv e^{t\mathcal{L}}$ is ***hypercontractive*** in $\mathbb{L}_q(\omega)$, $1 < q < \infty$ spaces iff

For any $1 < p_0 < p < \infty$

$$\exists T \in (0,\infty)\, \forall t > T \qquad ||\, P_t\, f\,||_p \leq ||\, f\,||_{p_0}$$

REM : Hypercontractivity in an interpolating family of Banach spaces $(\mathbb{B}_r)_{r\in I}$.

E.g. In $\mathbb{L}_p(d\lambda)$ or Orlicz spaces [BartheCatieauxRoberto].

Spectral Theory + Gaussian Bounds.

$$Hypercontractivity \iff ?\, Particle Structure?$$

-. Invariant Subspaces $\mathbb{L}_2(\mu) = \mathcal{H}_0 \oplus_{n\in\mathbb{N}} \mathcal{H}_n$

$$\mathcal{H}_j \perp \mathcal{H}_k, k \neq j, P_t(\mathcal{H}_n) \subset \mathcal{H}_n$$

-. Spectrum

$$\exists \varepsilon > 0 \quad \forall n \in \mathbb{N} \qquad \sigma(\mathcal{L} \restriction \mathcal{H}_n) \subset (-\infty, -\varepsilon\, n),$$

-. Gaussian Bounds: $\forall n \in \mathbb{N} \quad \forall f \in \mathcal{H}_n$

$$\exists C > 0 \qquad ||\, f\, ||_{\mathbf{4}} \leq C^n ||\, f\, ||_{\mathbf{2}},$$

E.g.s

- Free Quantum Field[Ne'66], [Si]...
- For Fermions[Gr'66], [CL'92]([Li'90])...
- 1-D Ising [B&Z'00]
- Product States on $NC\ \mathcal{A}$ & Weak Product Property [HO&Z'01], [B&Z'00]
- Quantum O-U [CaSa'08]
- Exotic CCR (q-OU [Biane'97], [Bozejko'99], [BozKuSp'97]; t-OU[Krolak'05].)
- Quasi-Free & Fermionic [TePaKa'14]

Conjecture : [OH&Z'01]

$\mathcal{A}_0 = M_{k\times k}, k \geq 2$, and $\mathcal{A}^{(n)} = \mathcal{A}_0^{\otimes n}$

$\text{Tr } f^2 \log f^2 - \text{Tr } f^2 \log \text{Tr } f^2 \leq c_{\text{opt}}(k) \sum_{i=1}^n \text{Tr}|\text{Tr}_i f - f|^2$

holds for any $f \in \mathcal{A}_{\text{sa}}^{(n)}$ with optimal constant

$c_{\text{opt}}(k) = (k/(k-2)) \log(k-1)$

Hypercontractivity for product states I.

- Product state $\omega \equiv \otimes_l \, \omega_{\Lambda_l}$, where $\omega_{\Lambda_l} \equiv \mathbf{Tr}_{\Lambda_l}(\rho_{\Lambda_l} \cdot)$

$$0 < \rho_{\Lambda_l} \leqq || \, \rho_{\Lambda_l} || < \infty, \; \Lambda_l \cap \Lambda_k = \emptyset \text{ for } k \neq l.$$

- $\mathbb{L}_p(\omega, s)$ norms

$$|| \, f \, ||^p_{\mathbb{L}_p(\omega,s)} \equiv \mathbf{Tr} \left| \rho_\Lambda^{\frac{1-s}{p}} \, f \, \rho_\Lambda^{\frac{s}{p}} \right|^p,$$

for $f \in \mathcal{A}_\Lambda$ with $\rho_\Lambda \equiv \prod_{\Lambda_l \cap \Lambda \neq \emptyset} \rho_{\Lambda_l}$.

- $\mathbb{L}_2(\omega, s)$ scalar product

$$< f \, , \, g >_{\mathbb{L}_2(\omega,s)} \equiv \mathbf{Tr} \Big(\rho_\Lambda^{\frac{1-s}{2}} \, f^* \, \rho_\Lambda^{\frac{s}{2}} \, g \Big) \, .$$

- Markov generator symmetric in $\mathbb{L}_2(\omega, s),\ \forall s \in [0,1]$,

  $$\mathcal{L} f \equiv \sum_{l \in \mathcal{R}} (E_{\Lambda_l}(f) - f)$$

  defined with

- Generalized Conditional Expectation

  $$E_X(f) \equiv \mathbf{Tr}_X(\xi^*_{\Lambda_l} f \xi_{\Lambda_l}) \ ,$$

  where for a bdd set $X \subset \mathcal{R}$,

$$\xi_X \equiv {\rho_{\Lambda_l}}^{\frac{1}{2}} (\mathbf{Tr}_X \rho_{\Lambda_l})^{-\frac{1}{2}}$$

Theorem :

- **Hypercontractivity** : The Markov semigroup $P_t \equiv e^{t\mathcal{L}}$ satisfies

  $|| P_t f ||_{\mathbb{L}_{p(t)}(\omega,s)} \leq || f ||_{\mathbb{L}_2(\omega,s)}$

  for any $s \in [0,1]$ with $p(t) \equiv 1 + e^{\alpha t}$, with some $\alpha > 0$.

- **Weak product property**:[Bodineau& Z'00], [Hebisz, Olkiewicz& Z'01]
  Therefore (for $s = \frac{1}{2}$)

  $\exists\, \tilde{c}_{\Lambda_0} \in (0,\infty)$
  $QEnt_2(f) \quad \leq \quad \tilde{c}_{\Lambda_0} \langle f, -\sum_{i \in \mathbb{Z}^d} (E_i(f) - f) \rangle_{\mathbb{L}_2(\omega,\frac{1}{2})}$

  where

$$QEnt_p(f) \equiv \lim_{\Lambda \to \mathcal{R}} \mathbf{Tr} \left| \rho_\Lambda^{1/2p} f \rho_\Lambda^{1/2p} \right|^p \left(\log \left| \rho_\Lambda^{1/2p} f \rho_\Lambda^{1/2p} \right| - 1/2p \log \rho_\Lambda \right).$$

Hypercontractivity and Spectral Gap

$(\mathbf{H}) \Rightarrow \|P_t f - \omega(f)\|_2^2 \leq e^{-\widetilde{m}t} \| f - \omega(f)\|_2^2$

Log-Sobolev Inequality Infinitesimal Condition for Hypercontractivity

Equivalence Theorem

Regular Dirichlet Forms

$\mathcal{E}_p$ are called $\mathbb{L}_p -$ regular iff $\exists d_0 \geq 0\, s.t. \forall\, q \geq 2$ and $g \in D(\mathcal{E}_p)$,

$$\mathcal{E}_2(I_{2,q}(g), I_{2,q}(g)) \leq \frac{q^2}{4(q-1)} \mathcal{E}_p(g, g) + d_0 \|g\|_q^q.$$

REM :

- $L_p(\mathrm{Tr})$ spaces. Then the corresponding generalized Dirichlet forms are L_p-regular.
- For subordinated semigroups $P_t(f) \equiv \int d\nu_t : \alpha_t(f)$ with: $\alpha_t(\cdot)$ denoting the modular automorphism.

Integration Lemma

Let $f \in \mathcal{A}^+ \cap D(\mathcal{E}_{q_0})$ be strictly positive

Then, with $q \equiv q(t) \equiv 1 + (q_0 - 1)e^{2t/c}$, we have

$$\frac{d}{\mathrm{dt}} \log \| P_t f \|_{q(t)} = \frac{\dot{q}_0}{\| f \|_q^q q_0} \left\{ \langle I_{p,q}(f), \mathbf{T}_q(f) \rangle - \| f \|_q^q \log \| f \|_q \right) - \frac{q c}{2(q-1)} c \mathcal{E}_p(f, f) \right\}$$

with $p \equiv p(t)$ complementary to $q \equiv q(t) \in (1, \infty)$.

Equivalence Theorem

Suppose P_t is a L_2-symmetric Feller semigroup which is hypercontractive, that is we have

$$\|P_t f\|_{q(t)} \leq \exp\{2d\left(\frac{1}{2}-\frac{1}{q(t)}\right)\|f\|_2 \qquad (*)$$

with $d\in[0,\infty)$ and $q(t)=1+e^{2t/c}$ defined with some constant $c\in(0,\infty)$.

Then the following Logarithmic Sobolev inequality is true.

$$\langle f, \mathbf{T}_2(f)\rangle - \|f\|_2^2 \log\|f\|_2 \leq c\mathcal{E}_2(f,f) + d\|f\|_2^2 \qquad (\mathbf{LS}(c,d))$$

with some constants $c\in(0,\infty)$ and $d\in[0,\infty)$ for all $f\in\mathbb{L}_2^+$ for which the r.h.s. is finite.

Conversely, if $\mathbf{LS}(c,d)$ inequality is true with some $c\in(0,\infty)$ and $d\in[0,\infty)$ and *additionally* $\mathcal{E}_p$ are $\mathbb{L}_p - regular$ with some constant $d_0\geq 0$, then $(*)$ holds with the constant d replaced by $d+c\cdot d_0$.

Poincare Inequality

$(\mathbf{LS}(c, d{=}0)) \Rightarrow \; m \, \| f - \omega(f) \|_2^2 \leq \mathcal{E}_2(f, f) \qquad \text{with } m \geq \frac{1}{2c}$

Rothaus Lemma

Let $\omega(f) = \mathrm{Tr}(\rho f)$. Suppose LS(c,d) inequality holds for every $f \in \mathcal{A}^+ \cap D(\mathcal{E}_2)$ and moreover that the following Spectral Gap inequality is true

$m \, \| f - \omega(f) \|_2^2 \leq \mathcal{E}_2(f, f)$

with some $m \in (0, \infty)$ independent of f. Then $\mathrm{LS}(c', d = 0)$ inequality holds with $c' \equiv c + (d+1)/m$.

Log-Sobolev as Orlicz-Sobolev Inequality

Orlicz functional : *For* $\gamma \geq 1$ *define*

$\mathfrak{N}_\rho(f) \equiv \mathrm{Tr}(|\rho^{\frac{1}{4}} f \rho^{\frac{1}{4}}|^2 (\log(\gamma\rho + ||\rho^{\frac{1}{4}} f \rho^{\frac{1}{4}}|^2) - \log\rho))$

Then $\exists c' \in (0, \infty)$

$\| f - \omega(f) \|_{\mathfrak{N}_\rho}^2 \leq c' \, \mathcal{E}_2(f, f)$

Optimal Product Property ??

Bounded Perturbation Lemma ??

LS(c) **for infinite dimensional models** ???

At least for classical interaction ?

1-D models ?

Exponential Bound and Distribution Tail estimates ???

$$|||f|||_{\mathrm{Lip}} < \infty \Rightarrow \text{“}\langle e^{\lambda f}\rangle\text{”} \leq e^{C\lambda^2 |||f|||^2_{\mathrm{Lip}} + \lambda\omega(f)}$$

Isoperimetric Functional Inequalities ???

$$\omega\Big(\sqrt{\mathcal{U}(f)^2 + c\Gamma(f)}\Big) \leq \mathcal{U}(\omega(f))$$

where $\mathcal{U}$ is concave on $[0,1]$ vanishing at the end points of the interval.

[Bo], [BL], [Z]$_{\text{For GibbsMeasures}}$

Hypercontractivity in Orlicz Spaces, Hamilton-Jacobi Eqn and Log-Sobolev ???

$\|Q_t(f)\|_{\exp} \le \|f\|_{\exp}$

where $\partial_t Q_t(f) = -\frac{1}{2}\Gamma(Q_t(f))$

$Q_t(f) = -\lim_{\varepsilon\to 0} \varepsilon \log P_{\varepsilon t} \exp\big(-\frac{1}{\varepsilon} f\big)$

[BGL],....[AIZ],[OZ]

Tallagrand Inequality ???

$\mathrm{dist}_2^2(\omega_1,\omega_2) \le \mathrm{Ent}(\omega_1|\omega_2)$

Quantum Bakry-Emery Scheme ???

Gradient Flow [Villani Otto] ***???***

Transportation Theory ???

Strong Ergodicity via Hypercontractivity ???

Equivalence of Complete Analyticity and Log-Sobolev Ineq ???

[SZ′92]

Slower tails weaker functional inequallities ???

Chellanging Computational Problems :

@ Large Interacting Systems & Slow Decay to Equilibrium

(Phase transitions, Disordered systems, Ground States,......)

Can Quantum Computing Say something about Quantum

Some References

• A.Guionnet & B.Zegarlinski, Lectures on Logarithmic Sobolev Inequalities,

Lecture Notes in Mathematics, Vol:1801, Pages:1-134 (2003) ISSN:0075-8434.

mathaa.epfl.ch/prst/mourrat/ihpin.pdf

• B.Zegarlinski, *Analysis of Classical and Quantum Interacting Particle Systems*,

pp 241 - 336, Quantum Interacting Particle Systems , Eds L. Accardi and F. Fagnola, World Sci 2002, *www.worldscientific.com/worldscibooks/10.1142/5055*

• Zegarlinski B, Analysis in Operator spaces, pp.121-139 in Quantum Stochastics and Information: Statistics, Filtering and Control,Eds V. P. Belavkin, M. Guta

• Olkiewicz R, Zegarlinski B, Hypercontractivity in noncommutative L-p spaces, *J. Funct.Analysis*, Vol: 161 (1999) 246-285,

• T. Bodineau and B. Zegarlinski, Hypercontractivity via spectral theory, **03**, 15 (2000). DOI: 10.1142/S0219025700000030

• M.Al-Rashed, B.Zegarlinski, Monotone norms and Finsler structures in noncommutative spaces, Infin. Dimens. Anal. Quantum. Probab. Relat. Top. Vol. 17, No. 04, 1450029 (2014)

Thank you for your attention !

Dziękuję za uwagę !

Merci !