Hypercontractivity and Log Sobolev inequalities in Quantum Information Theory

Patrick Hayden (Stanford University),
Christopher King (Northeastern University),
Ashley Montanaro (University of Bristol),
Mary Beth Ruskai (IQC, University of Waterloo)

2/23/2015–2/27/2015

1 Overview of the Field

Quantum Information Theory (QIT) is a highly interdisciplinary field, and many different areas of mathematics have played key roles in its development. Recently the topics of hypercontractivity (HC) and logarithmic Sobolev (LS) inequalities have found applications in a variety of problems within QIT, and this has led to a growth of interest among researchers in the field. The purpose of this workshop was to bring together researchers from both the 'traditional' areas of application and also the new areas of interest in QIT. Some talks were expository, some covered current areas of research, and some presented ideas for new directions of research.

Hypercontractivity, which concerns the contractive properties of semi-groups e^{-tH} from L^p to L^q, was introduced in Nelson's seminal 1965 work on semiboundedness of certain Hamiltonians arising in two-dimensional quantum field theory. In that context H was an operator on the (infinite dimensional) bosonic Fock space. The essential mathematical content was later extracted and reformulated in finite dimensional settings. There are two basic 'classical' versions, the continuous and discrete. The statement of hypercontractivity is the same in both cases: for $1 \leq p \leq q$, and for a suitable operator A,

$$\|e^{-tA}\|_{p\to q} = \sup_{\|f\|_p \leq 1} \|e^{-tA} f\|_q \leq 1 \quad \text{if and only if} \quad t \geq \frac{1}{2} \log\left(\frac{q-1}{p-1}\right)$$

The related concept of the logarithmic Sobolev inequality is an infinitesimal version of hypercontractivity, obtained by setting $p = 2$, $q(t) = 1 + e^{2t}$ and taking the derivative at $t = 0$. These methods and concepts have been generalized to the quantum setting (essentially by considering matrix spaces in place of function spaces), and then applied to a variety of problems in QIT. Some recent examples include: Kastoryano and Temme's extension of classical Markov chain results to estimate convergence rates for semigroups which arise in quantum Markov processes [**?**]; Gavinsky et al's use of hypercontractivity to give the first proof of exponential separation between one-way quantum and classical communication complexity of partial boolean functions [**?**]; and Klartag and Regev's use of hypercontractivity on the n-sphere to resolve the long-standing conjecture that one-way quantum communication is exponentially stronger than even two-way classical communication [**?**].

2 Logistics and organization

The schedule of the workshop was arranged with four lectures per day, in order to allow time for discussions and follow-up. The success of this workshop was due, in part, to a decision to focus on a particular area of mathematics of potential importance to quantum information and to bring together people with very different backgrounds in mathematics and in the physical sciences. Several participants commented that they found the focused nature of the workshop particularly inspiring and productive. The intensive nature of a BIRS workshop naturally encouraged interaction and new collaborations. An important feature of the workshop was a session devoted to the presentation and discussion of open problems. This led to discussions and collaborations, and several preprints as described below.

3 Major topics of discussion

3.1 Overviews and historical surveys

William Beckner provided a survey talk on LS and HC, describing the context of the original work of Gross and others, and connected the different formulations of LS for Gauss measure on $\mathbf{R}^n$ and uniform measure on the sphere. Chris King described the origins of HC in Nelson's work on semi-boundedness of the Hamiltonian for the interacting ϕ^4 field theory in two space-time dimensions. Eric Carlen reviewed the work of Gross, and Ball, Carlen, Lieb on HC for fermions, and described the important links between this work and the subsequent proofs of uniform smoothness and uniform convexity for non-commutative L^p spaces. Ronald de Wolf described the formulation of HC for functions on the Boolean cube, and reviewed inportant applications of this notion in computer science, including the famous KKL theorem. Boguslaw Zegarlinski described the extensions of HC and LS to noncommutative spaces. These concepts, dating back to the late 90s, have recently come to prominence in QIT, being used for example by Kastoryano and Temme to prove bounds on the mixing time of quantum Markov processes. Ashley Montanaro provided an overview of recent applications of HC and LS in quantum information theory, including bounds for the bias of lcoal games, separation of quantum and classical communication complexity, and limitations on quantum random access codes.

3.2 Mixing and Markov chains

One of the principal applications of LS inequalities is to bound the mixing times of classical Markov chains. Using the methodology of Olkiewicz and Zegarlinski, Kastoryano and Temme extended the classical results to one-parameter semigroups of quantum channels, and obtained bounds for mixing times in terms of LS constants. The extension to the most general case would require the property of strong L^p regularity, which is an open problem in the general case. For classical Markov chains there is an easy 'additivity' property for LS constants which implies that the mixing time of independent copies of a chain is the same as for a single copy. The corresponding additivity result has not been demonstrated for quantum channel semigroups, and is suspected to fail in general. This failure would prevent an easy derivation of the mixing time bound for multiple copies of a channel semigroup.

Fernando Brandao discussed the quantum version of a property of thermal models in statistical mechanics, namely the equivalence of mixing in time (size-independent spectral gap) and mixing in space (bounded correlation length). For commuting Hamiltonians the equivalence goes through, however in general the relation is not so clear. David Perez-Garcia discussed the relation between how the mixing time scales with the system size, and the area law. The area law is a conjectured property of quantum systems, namely that the entanglement of the ground state grows with the size of the boundary rather than the bulk. Oleg Szehr reported on a new approach to estimating mixing times for quantum channel semigroups. This approach yields spectral bounds on norms of functions of transition maps of (classical or quantum) Markov processes and implies new estimates for the convergence of such processes to stationarity. The main technical ingredient is the use of a Wiener algebra functional calculus in the context of bounded semigroups.

3.3 Nonlocal games and complexity

A productive way to explore the power of quantum correlations has been via the concept of nonlocal games. These are games where two separated parties, who are not allowed to communicate, aim to each output some value which depends on their joint inputs. An especially interesting class of nonlocal games is XOR games, where whether the players succeed depends only on the sum modulo 2 of their outputs. Quantum correlations are known to yield advantages over classical correlations in XOR games, one example being the famous CHSH game. As well as their practical importance as experimental tests of quantum mechanics, these games have a number of links to different areas of mathematics, and in particular are closely connected to Grothendieck inequalities. Harry Buhrman presented an overview of the evolution of XOR games and their applications. These include a recent resolution of a 35-year-old problem of Varopoulos in operator algebras, showing that the space of compact operators on a Hilbert space is a Q-algebra under Schur product. Thomas Vidick described recent advances in generalisations of XOR games to the setting where the questions to the players are quantum, including some intriguing connections to noncommutative Grothendieck inequalities.

Other presentations described work concerned with different notions of quantum complexity. In the computational setting, SDP hierarchies are a rather canonical approach to find relatively efficient algorithms for optimization problems in quantum information. Aram Harrow reviewed this notion of SDP hierarchy and described work showing that SDP hierarchies can sometimes be replaced with nets. This can be a significant simplication (conceptually and algorithmically), and also provides insight into why the class of 1-LOCC measurements is "easier" to optimise over than general measurements. In information theory, hypercontractivity has been used classically to prove limitations on transformation of nonlocal correlations, via the concept of hypercontractivity ribbon due to Ahlswede and Gács. Salman Beigi described his work with Delgosha on a generalisation of this concept to the noncommutative setting in order to put bounds on the asymptotic transformation of quantum correlations.

3.4 Links to other research areas

Several speakers described new directions of research with connections to other areas of mathematics and physics. Reinhard Werner discussed how to combine classical and quantum harmonic analysis via a direct sum construction. Mathilde Perrin described new results on hypercontractivity for group von Neumann algebras. Michael Kastoryano presented results on quantum Gibbs samplers. A Gibbs sampler is a Markov process which drives a system to thermal equilibrium (its Gibbs state). Recent work has built on the theory of mixing times of quantum channels to show that, for a number of physical systems of interest, Gibbs samplers provide an efficient way of preparing the associated Gibbs states on a quantum computer.

4 Research resulting from the workshop

Discussions at the workshop led to several research collaborations and results. In the open problem session Ashley Montanaro posed the question of proving a reverse hypercontractivity result in QIT. Subsequently Toby Cubitt, Michael Kastoryano, Ashley Montanaro and Kristan Temme solved this problem, leading to the preprint 'Quantum reverse hypercontractivity' (arXiv:1504.06143) which also contains several applications of their results. Another collaboration between Aram Harrow and Ashley Montanaro led to the preprint 'Extremal eigenvalues of local Hamiltonians' (arXiv:1507.00739). Also Aram Harrow made important progress on his paper 'Approximate orthogonality of permutation operators' at the workshop. Salman Beigi and Chris King started a collaboration on the problem of determining the form of HC and LS inequalities for the completely bounded norms.

References

[1] D. Gavinsky, J. Kempe, I. Kerenidis, R. Raz, and R. de Wolf, Exponential separations for one-way quantum communication complexity, with applications to cryptography, *Proc. 39th Annual ACM Symp. Theory of Computing* (2007), 516–525. quant-ph/0611209.

[2] M. J. Kastoryano and K. Temme, Quantum logarithmic Sobolev inequalities and rapid mixing, *J. Math. Phys.* **54** (2013), 052202 . arXiv:1207.3261.

[3] B. Klartag and O. Regev, Quantum one-way communication can be exponentially stronger than classical communication, *Proc. 43rd Annual ACM Symp. Theory of Computing* (2011), 31–40. arXiv:1009.3640.