Topological dynamics of the automorphism groups of Hrushovski constructions

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Joint project with Jan Hubička and Jaroslav Nešetřil

Background

- Question asked (around 2011) by: Bodirsky, Pinsker, Tsankov; Nešetřil; Nguen van Thé:
  - If M is countable ω-categorical, is there an ω-categorical expansion N of M with $\mathrm{Aut}(N)$ extremely amenable?
- Particularly interesting case: M homogeneous in a finite relational language.
- Why ask the question?
  - Ubiquity of ω-categorical structures with e.a. automorphism groups
  - Applications: reducts; complexity of CSP's
  - Describing $M(G)$ for G closed, oligomorphic permutation group.
  - Evidence. Work on Ramsey expansions of Fraïssé classes: Nešetřil - Rödl; Jasinski, Laflamme, Nguen van Thé, Woodrow; ...

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Main results

THEOREM: There is a countable, ω-categorical structure M with the property that if $H \leq \mathrm{Aut}(M)$ is extremely amenable, then H has infinitely many orbits on M^2. In particular, there is no ω-categorical expansion of M whose automorphism group is extremely amenable.

Different formulation, using Kechris, Pestov, Todorcevic correspondence:

THEOREM: There is a Fraïssé class $\mathcal{C}$ with finitely many isomorphism types of each finite size which has no precompact Ramsey expansion.

Using Zucker's work:

COROLLARY: There is a closed, oligomorphic permutation group $G \leq S_\infty$ whose universal minimal flow $M(G)$ is not metrizable.

REMARK: Question is open for M homogeneous in a finite relational language.

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The example

Hrushovski construction (1988)

FACT: (Hrushovski) There is a countable, ω-categorical graph $(M; R)$ in which all vertices have infinite valency and with the property that for every finite subset A of M we have $|R[A]| \leq 2|A|$.

Note: $R[A]$ denotes $R \cap [A]^2$.

DEF: A graph $(X; R)$ is 2-*orientable* if it is the undirected reduct of a directed graph $(X; D)$ with the property that the out-valency of every vertex is at most 2. Say $(X; D)$ here is a 2-*orientation* of $(X; R)$.

LEMMA: A graph $(X; R)$ is 2-orientable if and only if for every finite $A \subseteq M$ we have $|R[A]| \leq 2|A|$. □

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Proof of Theorem

- Let $(M; R)$ be as in the Fact; $G = \mathrm{Aut}(M)$. Suppose $H \leq G$ is e.a.
- Let $X \subseteq \{0,1\}^{M^2}$ be the space of 2-orientations of $(M; R)$.
- Then
  - X is non-empty (Lemma)
  - X is closed in $\{0,1\}^{M^2}$, so is compact
  - X is G-invariant
- So H preserves a 2-orientation D of $(M; R)$
- $H \leq \mathrm{Aut}(M; D)$ has infinitely many orbits on M^2. Otherwise, there is a bound k on the size of finite H_a-invariant sets for $a \in M$. Take a, k optimal here. Find $c \to a$. Then there is a finite H_c-invariant set of size $\geq k + 1$. Contradiction.

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Generalisation

THEOREM: Suppose M is a countable, ω-categorical structure and $R \subseteq [M]^n$ is a non-empty, $\mathrm{Aut}(M)$-invariant relation with the properties:

1. For every $a \in M$, there are either zero or infinitely many $r \in R$ with $a \in r$;
2. There is some $N \in \mathbb{N}$ such that for all finite $A \subseteq M$ we have $|R[A]| \leq N|A|$.

Then every extremely amenable subgroup H of $\mathrm{Aut}(M)$ has infinitely many orbits on M^2.

Hrushovski constructions

- $\mathcal{C}$: class of finite graphs $(A; R)$
- If $C \subseteq A \in \mathcal{C}$ let
$$\delta(C) = 2|C| - |R[C]|.$$
(Predimension of C.)
- If $A \subseteq B \in \mathcal{C}$ write $A \leq_d B$ if $\delta(X) > \delta(A)$ whenever $A \subset X \subseteq B$.
- Write $A \leq_s B$ if $\delta(X) \geq \delta(A)$ whenever $A \subset X \subseteq B$.
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The ω-categorical case.

- $F : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ an increasing function which tends to infinity.
- Let
$$\mathcal{C}_F = \{A \in \mathcal{C} : \delta(Y) \geq F(|Y|) \text{ for all } Y \subseteq A\}.$$

- For suitable F the class $(\mathcal{C}_F, \leq_d)$ has free amalgamation over $\leq_d$-substructures.
- In this case there is a countable generic structure M_F for the class characterised by:
  - M_F is the union of a chain of finite $\leq_d$-subgraphs;
  - every graph in $\mathcal{C}_F$ is isomorphic to a $\leq_d$-subgraph of M_F;
  - isomorphisms between finite $\leq_d$-subgraphs of M_F extend to automorphisms.
- The structure M_F is ω-categorical.

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- Let $\mathcal{C}_0 = \{A \in \mathcal{C} : \emptyset \leq_s A\}$.
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PROBLEMS: (1) Look at minimal subflows for $\mathrm{Aut}(M_F)$ and $\mathrm{Aut}(M_0)$ acting on the space of 2-orientations of M_F and M_0 respectively.

(2) Look for good Ramsey expansions of $\mathcal{C}_0, \mathcal{C}_F$.

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Another variation

- $\mathcal{C}_1 \subseteq \mathcal{C}_0$ consists of finite graphs having a 2-orientation without a directed cycle.
- If $A \subseteq B \in \mathcal{C}_1$ write $A \leq_1 B$ if there is a 2-orientation of B without directed cycles in which A is closed under out-vertices.
- $(\mathcal{C}_1, \leq_1)$ is a free amalgamation class; call the generic structure M_1.

Consider G_1 acting on the compact space X_1 of directed-cycle-free 2-orientations of M_1.

THEOREM:

1. X_1 is a minimal G_1-flow.
2. Every G_1-orbit is meagre in X_1.

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Proofs (1)

(1) Let $\mathcal{D}_1$ be the finite directed-cycle-free digraphs with at most 2 out-edges at each vertex. Minimality of X_1 follows from:

EXPANSION PROPERTY: If $A \in \mathcal{C}_1$ there is $A \leq_1 B \in \mathcal{C}_1$ such that whenever $\overline{A}$ and $\overline{B}$ are expansions of A, B in $\mathcal{D}_1$, there is a $\leq_1$-embedding of $\overline{A}$ into $\overline{B}$.

Proofs (2)

(2) Let $t \in X_1$. If $a \in M_1$ let $\mathrm{cl}^{M_1,t}(a)$ be the out-closure of a in M_1.

For suitable $a \in M_1$ show $G_a \cdot t$ is nowhere dense in X_1.

Suppose not. Then there is $a \in B \subseteq_{finite} M_1$ and $(B, t_B) \in \mathcal{D}_1$ such that

- if $B \subseteq C \subseteq_{finite} M_1$ and $(C, t_C) \in \mathcal{D}_1$ extends (B, t_B) there is $g \in G_a$ such that (M_1, gt) extends (C, t_C).

Thus $\mathrm{cl}^{C,t_C}(a)$ embeds (via g^{-1}) into $\mathrm{cl}^{M_1,t}(a)$.

So if $s \in X_1$ extends t_B then $\mathrm{cl}^{M_1,s}(a)$ embeds into $\mathrm{cl}^{M_1,t}(a)$.

From this, one can derive a contradiction (too many possibilities for s).

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Thus $\mathrm{cl}^{C,t_C}(a)$ embeds (via g^{-1}) into $\mathrm{cl}^{M_1,t}(a)$.

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From this, one can derive a contradiction (too many possibilities for s).

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In progress

- Try to make this work for X_0 and X_F.
- Look for Ramsey expansions of the classes $(\mathcal{C}_0, \leq_s)$ and $(\mathcal{C}_F, \leq_d)$ which have EP.