Fraïssé categories and their applications

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Motivations, goals

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- Exploring the crucial points of Fraïssé theory

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- Exploring the crucial points of Fraïssé theory
- Developing tools for constructing / recognizing objects with high level of homogeneity

The setup

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(A2) For every $X = \lim \vec{x} \in \mathrm{Obj}(\mathfrak{L})$, $y \in \mathrm{Obj}(\mathfrak{S})$, for every arrow $f \colon y \to X$ there exists n such that $f = x_n^\infty \circ f'$ for some $f' \in \mathfrak{S}$.

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For every category $\mathfrak{S}$ there exists a category $\sigma\mathfrak{S}$ such that $\langle \mathfrak{S}, \sigma\mathfrak{S} \rangle$ satisfies (A1), (A2).

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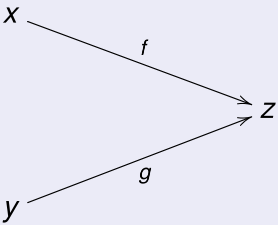
- The objects of $\sigma\mathfrak{S}$ are sequences (i.e., covariant functors) of type $\mathbb{N} \to \mathfrak{S}$.
- The $\sigma\mathfrak{S}$-arrows are natural transformations into subsequences.

Fraïssé categories – the discrete case

Definition

We say that $\mathfrak{S}$ is directed if for every $x, y \in \mathrm{Obj}(\mathfrak{S})$ there exist $z \in \mathrm{Obj}(\mathfrak{S})$ and $\mathfrak{S}$-arrows $f\colon x \to z$, $g\colon y \to z$.



Definition

We say that $\mathfrak{S}$ has the **amalgamation property** if for every $\mathfrak{S}$-arrows $f\colon z \to x$, $g\colon z \to y$ there exist $\mathfrak{S}$-arrows $f'\colon x \to w$, $g'\colon y \to w$ such that the diagram

$$\begin{array}{ccc} y & \xrightarrow{\;g'\;} & w \\ {\scriptstyle g}\uparrow & & \uparrow{\scriptstyle f'} \\ z & \xrightarrow[\;f\;]{} & x \end{array}$$

is commutative.

Domination

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Let $\mathfrak{F}$ be a subcategory of $\mathfrak{S}$. We say that $\mathfrak{F}$ is dominating in $\mathfrak{S}$ if the following conditions are satisfied.

(D1) For every $x \in \mathrm{Obj}(\mathfrak{S})$ there exists an $\mathfrak{S}$-arrow $f\colon x \to y$ such that $y \in \mathrm{Obj}(\mathfrak{F})$.

(D2) Given an $\mathfrak{S}$-arrow g with $\mathrm{dom}(g) \in \mathrm{Obj}(\mathfrak{F})$, there exists an $\mathfrak{S}$-arrow h such that $h \circ g \in \mathfrak{F}$.

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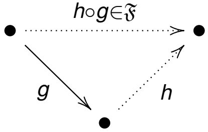
(D2) 

Main definition

Definition

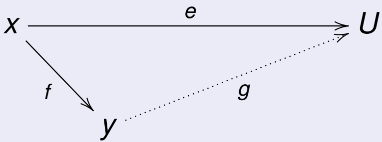
We say that $\mathfrak{S}$ is a Fraïssé category if

- $\mathfrak{S}$ is directed,
- $\mathfrak{S}$ has the amalgamation property,
- $\mathfrak{S}$ is dominated by a countable subcategory.

Theorem (Droste & Göbel 1993, K. 2007)

Assume $\mathfrak{S}$ is a Fraïssé category. Then there exists a unique, up to isomorphism, object $U \in \mathrm{Obj}(\mathfrak{L})$ with the following properties:

1. *For every $x \in \mathrm{Obj}(\mathfrak{S})$ there exists an $\mathfrak{L}$-arrow $e \colon x \to U$.*
2. *For every $e \colon x \to U$ with $x \in \mathrm{Obj}(\mathfrak{S})$, for every $\mathfrak{S}$-arrow $f \colon x \to y$ there exists an $\mathfrak{L}$-arrow $g \colon y \to U$ such that $e = g \circ f$.*



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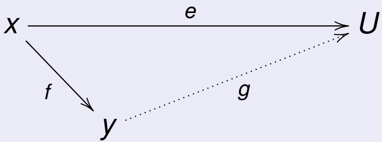


Definition

We call U the Fraïssé limit of $\mathfrak{S}$ and write $U = \mathrm{Flim}(\mathfrak{S})$.

Important features of Fraïssé limits

Theorem (Universality)

Let $U = \mathrm{Flim}(\mathfrak{S})$*. Then for every* $X \in \mathrm{Obj}(\mathfrak{L})$ *there exists an* $\mathfrak{L}$*-arrow* $e \colon X \to U$*.*

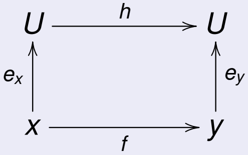
Important features of Fraïssé limits

Theorem (Universality)

Let $U = \mathrm{Flim}(\mathfrak{S})$. Then for every $X \in \mathrm{Obj}(\mathfrak{L})$ there exists an $\mathfrak{L}$-arrow $e\colon X \to U$.

Theorem (Homogeneity)

Let $U = \mathrm{Flim}(\mathfrak{S})$. For every $\mathfrak{S}$-arrow $f\colon x \to y$, for every $\mathfrak{L}$-arrows $e_x\colon x \to U$, $e_y\colon y \to U$ there exists an automorphism $h\colon U \to U$ satisfying $h \circ e_x = e_y \circ f$.



Some examples

Example

Let $\mathfrak{S}$ be a category of finitely generated models of a fixed first-order language, $\mathfrak{L}$ a suitable category of countably generated structures. If $\mathfrak{S}$ is hereditary, then $\mathrm{Flim}(\mathfrak{S})$ is the same as the Fraïssé limit in the model-theoretic sense.

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Then $\mathrm{Flim}(\mathfrak{S})$ is the Cantor set.

Projective Fraïssé classes

Example (Irwin & Solecki 2006)

Let $\mathfrak{S}$ be a class of finite nonempty structures of some fixed first-order language. Turn it into a category, by saying that f is an arrow from x to y if $f\colon y \to x$ is an epimorphism.

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Then $\mathfrak{S}$ is a Fraïssé category $\iff$ $\mathfrak{S}$ is a projective Fraïssé class in the sense of Irwin & Solecki.

Posets

Example

Let $\langle S, \leqslant \rangle$ be a poset. Then S is a category in which arrows are of the form $\langle x, y \rangle$ with $x \leqslant y$.

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Let $\langle S, \leqslant \rangle$ be a poset. Then S is a category in which arrows are of the form $\langle x, y \rangle$ with $x \leqslant y$.
Amalgamation is equivalent to directedness. Thus, $\langle S, \leqslant \rangle$ is Fraïssé $\iff \langle S, \leqslant \rangle$ is up-directed and has countable cofinality.

Monoids

Example

Let $\langle S, \circ \rangle$ be a monoid (i.e. a semigroup with a unit), viewed as a category. It is automatically directed. Amalgamation means

$$(\forall\, x, y \in S)(\exists\, x', y' \in S)\ \ x' \circ x = y' \circ y.$$

This holds, for example, when $\langle S, \circ \rangle$ is commutative.
Note that

$$T \subseteq S \text{ is dominating} \iff (\forall\, x \in S)(\exists\, y \in S)\ \ y \circ x \in T.$$

Fraïssé categories – the continuous case

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This means that each hom-set $\mathfrak{L}(X, Y)$ has a metric $\varrho = \varrho_{X,Y}$ such that

1. $\varrho(f \circ g_1, f \circ g_2) \leqslant \varrho(g_1, g_2)$
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(A2) If $X = \lim \vec{x}$, where $\vec{x}$ is a sequence in $\mathfrak{S}$, then for every $\mathfrak{L}$-arrow $f \colon y \to X$, for every $\varepsilon > 0$ there exist n and an $\mathfrak{S}$-arrow $f' \colon y \to x_n$ such that $\varrho(x_n^\infty \circ f', f) < \varepsilon$.

Domination revisited

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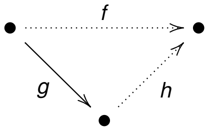

Definition

We say that $\mathfrak{S}$ has the almost amalgamation property if for every $\mathfrak{S}$-arrows $f\colon z \to x$, $g\colon z \to y$, for every $\varepsilon > 0$ there are $\mathfrak{S}$-arrows $f'\colon x \to w$, $g'\colon y \to w$ such that

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Definition

We say that $\mathfrak{S}$ is a Fraïssé category if it is directed, countably dominated and has the almost amalgamation property.

Theorem

Let $\mathfrak{S}$ be a Fraïssé category. There exists a unique, up to isomorphism, $\mathfrak{L}$-object U satisfying

1. *For every $x \in \operatorname{Obj}(\mathfrak{S})$ there exists an $\mathfrak{L}$-arrow $e\colon x \to U$.*
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Remark

The Urysohn space is homogeneous with respect to finite sets, while the Gurariĭ space is not homogeneous with respect to finite-dimensional spaces.

The pseudo-arc

Example

Let $\mathfrak{S}$ be the category whose objects are closed intervals $[0, n]$ ($n \in \mathbb{N}$) and arrows are non-expansive surjections. More precisely, $f \in \mathfrak{S}([0, n], [0, m])$ iff f is a non-expansive surjection from $[0, m]$ onto $[0, n]$.

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Fact

$\mathfrak{S}$ is a Fraïssé category, although it fails the amalgamation property.

$\mathfrak{L}$ is the category of all nonempty *chainable continua* (a continuum = a compact metrizable connected space).
The Fraïssé limit of $\mathfrak{S}$ is the *pseudo-arc*.

The Poulsen simplex

Example (from joint work with A. Kwiatkowska)

Let $\mathfrak{S}$ be the category whose objects are finite-dimensional simplices, while an arrow from Δ_n to Δ_m is an affine retraction from Δ_m onto $\Delta_n \subseteq \Delta_m$ (here $n \leqslant m$).

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Claim

$\mathfrak{S}$ is a Fraïssé category. Its Fraïssé limit is the Poulsen simplex.

The Lelek fan

Example (from joint work with A. Kwiatkowska)

Let $\mathfrak{S}$ be the category whose objects are of the form

$$\Delta(S) = (S \times [0,1])/_{\sim},$$

where S is a finite set and $\sim$ collapses $\{(s,0) \colon s \in S\}$ to a point 0, called the *vertex* of $\Delta(S)$. Sets of the form $\{s\} \times [0,1]$ are called *spikes* in $\Delta(S)$.

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An $\mathfrak{S}$-arrow from $\Delta(S)$ to $\Delta(T)$ with $S \subseteq T$ is a retraction $r \colon \Delta(T) \to \Delta(S)$ which is affine on spikes and fixes the vertex.

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Claim

$\mathfrak{S}$ is a Fraïssé category and its Fraïssé limit is the Lelek fan.

Bad news

Fact

The category of finite metric spaces with isometric embeddings is not countably dominated.

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Fact

The category of finite-dimensional Banach spaces with linear isometric embeddings is not countably dominated.

Fact

A separable Banach space G is linearly isometric to the Gurariĭ space if and only if

(G) *For every finite-dimensional spaces $X \subseteq Y$, for every linear isometric embedding $e\colon X \to G$, for every $\varepsilon > 0$ there exists an ε-isometric embedding $f\colon Y \to G$ such that $\|f \restriction X - e\| < \varepsilon$.*

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Measured categories

Measures

Definition

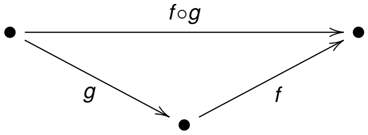
A measure on a category $\mathfrak{K}$ is a function $\mu\colon \mathfrak{K} \to [0, +\infty]$ satisfying the following conditions:

(M1) $\mu(\mathrm{id}_x) = 0$ for every object x.

(M2) $\mu(f \circ g) \leqslant \mu(f) + \mu(g)$ whenever $f \circ g$ is defined.

(M3) $\mu(g) \leqslant \mu(f \circ g) + \mu(f)$ whenever $f \circ g$ is defined.

A pair $\langle \mathfrak{K}, \mu \rangle$ will be called a measured category.



Example

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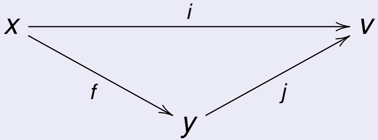
Example

Let $\mathfrak{K} = \langle X, X \times X \rangle$ be a quasi-ordered set, treated as a category such that $\mathfrak{K}(x, y) = \{\langle x, y \rangle\}$ for every $x, y \in X$. Then a measure on $\langle X, \leqslant \rangle$ is a pseudo-metric (we allow 0 for distinct points).

We assume that $\mathfrak{S}$ is a measured category enriched over metric spaces.

Axiom (T)

For every $\varepsilon > 0$ there is $\delta > 0$ such that whenever $f\colon x \to y$ satisfies $\mu(f) < \delta$ then there exist $i\colon x \to v$, $j\colon y \to v$ such that $\mu(i) = \mu(j) = 0$ and $\varrho(i, j \circ f) < \varepsilon$.



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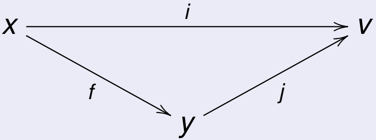

Lemma (Solecki & K. 2013)

The category of finite-dimensional Banach spaces satisfies this axiom (with $\delta = \varepsilon$).

Typical examples of measures

Example

Let $\mathfrak{K}$ be a metric category, $\mathfrak{K}_0 \subseteq \mathfrak{K}$ a subcategory with the same objects. Define

$$\mu(f) = \inf\{\varrho(i, j \circ f) \colon i, j \in \mathfrak{K}_0, \text{ and } \varrho(i, j \circ f) \text{ is defined } \}$$

Then μ is a measure on $\mathfrak{K}$ satisfying (T).

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Let $\mathfrak{K}$ be the category of Banach spaces, $\mathfrak{K}_0$ the subcategory of all isometric embeddings. Then $\mu(f) < \varepsilon \leqslant 1 \iff f$ is an ε-isometry.

Typical examples of measures

Example

Let $\mathfrak{K}$ be a metric category, $\mathfrak{K}_0 \subseteq \mathfrak{K}$ a subcategory with the same objects. Define

$$\mu(f) = \inf\{\varrho(i, j \circ f) \colon i, j \in \mathfrak{K}_0, \text{ and } \varrho(i, j \circ f) \text{ is defined }\}$$

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More details:

W. Kubiś, *Metric-enriched categories and approximate Fraïssé limits*, preprint, `http://arxiv.org/abs/1210.6506`.

After adapting the other assumptions and axioms, we obtain the final notion of a Fraïssé category.

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Theorem

The Urysohn space is the Fraïssé limit of the category of finite metric spaces.

Theorem

The Gurariĭ space is the Fraïssé limit of the category of finite-dimensional Banach spaces.

More examples

Theorem (Garbulińska-Węgrzyn & K. 2015)

There exists a non-expansive linear operator $\Omega\colon \mathbb{G} \to \mathbb{G}$ such that every non-expansive linear operator acting between separable Banach spaces is isometric to a restriction of Ω.
The operator Ω has certain extension property, making it isometrically unique.

Quasi-Banach spaces

Definition

Let $0 < p \leqslant 1$. A p-norm is a function $\|\cdot\|$ satisfying the usual axioms of a norm, except that the triangle inequality is replaced by

$$\|x + y\|^p \leqslant \|x\|^p + \|y\|^p.$$

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- F. Cabello Sánchez, J. Garbulińska-Węgrzyn, W. Kubiś, *Quasi-Banach spaces of almost universal disposition*, Journal of Functional Analysis 267 (2014) 744–771.

Theorem

Let V be a separable p-Banach space, where $0 < p \leqslant 1$. Then there exists a non-expansive linear operator $P_V \colon U_V \to V$ with the following properties:

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Fact

If $p = 1$ and $V = \mathbb{G}_p$ then $P_V = \Omega$.

Further examples

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