Homogeneous Structures

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1 Overview of the Field

A relational first order structure is homogeneous if every isomorphism between finite substructures extends to an automorphism. Familiar examples of such structures include the rational numbers with the usual order relation, the countable random and so called Rado graph, and many others. Countable homogeneous structures arise as Fraïssé limits of amalgamation classes of finite structures, and have connections to model theory, permutation group theory, combinatorics (for example through combinatorial enumeration, and through Ramsey theory), and descriptive set theory. Our principal objective at this workshop was to promote close interactions between different fields of mathematics affected by recent developments related to homogeneous structures, including researchers in the areas of combinatorics, descriptive set theory, dynamical systems, group theory, metric spaces, and model theory.

Some of the mainstream recent themes that have emerged include the following.

- Universal objects: Examples include the Fraïssé theory in logic and generalizations in model theory, universal graphs in combinatorics, the universal Urysohn space in topology.
- Homogeneous structures: Automorphism groups of homogeneous structures, Set Homogeneity, Polish groups and topological dynamics, structural Ramsey theory, constraint satisfaction, omega-categoricity and amalgamation constructions, metric homogeneous structures, and classification results.

Universal objects are central to Mathematics in a sense that they may reflect properties and non-properties of a given class of structures. They are typically very homogeneous, and hence the deep connection between these areas. Recent applications across the above themes and disciplines provide a unique opportunity to gather experts with knowledge from various mathematical angles, from model theorists who provide techniques for constructing such objects, to permutation group theorists who can provide insight into the automorphism groups of these rich structures.

2 Recent Developments and Open Problems

While the initial focus of the area was on the classification of various homogeneous structures and their Ramsey properties, the subject has since widened extensively to include connections to infinite permutation groups, dynamical systems through questions about continuous actions of Polish groups on compact spaces, making extensive use of other tools such as Ramsey classes. In other directions, versions of constraint satisfaction problems were established through infinite (usually homogeneous) structures, and Fraïssé limits have been extended to metric structures.

These various developments are well exemplified by the variety of open problems which arose at the workshop, a few of them listed here.

2.1 Itaï Ben Yaacov, Université Claude Bernard

Problem: Let G be a Polish Roelcke precompact group with compatible left invariant metric d_L. Define a new metric as $d_u(g,h) = \sup_{k\in G} d_L(gk, hk)$. Is there any relationship between discreteness of d_u and the fact that G cannot act transitively by isometries on a complete metric space?

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2.2 Gabriel Conant, University of Notre-Dame

Problem: Suppose that S is an infinite subset of positive reals that has the 4-value condition, as defined in [1]. Fix $A \subset S$ finite. Is there a finite S_0 with the 4-value condition such that $A \subset S_0 \subset S$?

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2.3 Cameron Freer, Gamalon Labs

Call a relational structure *highly homogeneous* when for every k, its automorphism group acts transitively on the k-element sets. Peter Cameron classified the countable highly homogeneous relational structures in 76, and all of them turn out to satisfy the strong amalgamation property.

Problem: Is there an elementary proof of this latter fact, without referring to the classification?

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2.4 Robert Gray, University of East Anglia

A relational structure M is *set-homogeneous* if whenever two finite substructures U and V are isomorphic, there is an automorphism $g \in \mathrm{Aut}(M)$ such that $g(U) = V$. As an example, consider the countably infinite graph $R(3)$ defined as follows. Its vertex set is a countable dense subset of the unit circle with no two points make an angle of $2\pi/3$ at the centre of the circle. Its edge set is given by $x \sim y$ iff $0 < \arg(x/y) < 2\pi/3$.

Problem 1: [from [DGMS94]] Are $R(3)$ and its complement $\overline{R(3)}$ the only countable set-homogeneous graphs which are not homogeneous?

Problem 2: [from [GMPR12]] Is there a countable set-homogeneous tournament that is not homogeneous?

Related to this, there is also the problem:

Problem 3: Classify the countably infinite set-homogeneous digraphs.

A structure (with an age which has finitely many n-element structures up to isomorphism) is *homogenizable* if it can be made homogeneous by adding finitely many relations to the language (without changing the automorphism group).

Problem 4: [from [DGMS94]] Is there a set-homogeneous structure which is not homogenizable?

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2.5 Jordi Lopez-Abad, Instituto de Ciencias Matemáticas

Problem 1: [from [1]] Does the dual Ramsey theorem hold for equidistributed partitions?

Problem 2: Let $\mathbb{G}$ be the Gurarij space, and $\mathrm{iso}(\mathbb{G})$ be its linear isometry group. Is $\mathrm{iso}(\mathbb{G})$ compactly approximable, ie does it admit an increasing chain of compact subgroups whose union is dense? Is $\mathrm{iso}(\mathbb{G})$ Lévy?

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2.6 Martino Lupini, California Institute of Technology

Problem 1: [from [4]] Let $\mathbb{G}$ be the Gurarij space, and $\mathrm{iso}(\mathbb{G})$ be its linear isometry group. Does it have the automatic continuity property, ie is every algebraic homomorphism from $\mathrm{iso}(\mathbb{G})$ to a separable group continuous?

Problem 2: Given a Fraisse class (or a metric Fraisse class) $\mathcal{C}$ with limit M, is there a natural assumption on $\mathcal{C}$ that guarantees that for every countable substructure X of M, $\mathrm{Aut}(X)$ embeds in $\mathrm{Aut}(M)$ via some embedding j of X in M so that every automorphism of $j(X)$ extends to an automorphism of M?

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2.7 Micheal Pawliuk, University of Toronto

Problem: Let $\mathbb{S}$ denote the semi-generic digraph. Is $\mathrm{Aut}(\mathbb{S})$ uniquely ergodic, ie is there a unique invariant Borel probability measure whenever $\mathrm{Aut}(\mathbb{S})$ acts continuously and minimally on a compact Hausdorff space? Equivalently, is there a unique invariant Borel probability measure on the universal minimal flow of $\mathrm{Aut}(\mathbb{S})$?

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2.8 Michael Pinsker, Charles University

Problem 1: Let $F \subset \omega^{\omega^3}$ be topologically closed, closed under composition, and containing the set $\{\pi_1, \pi_2, \pi_3\}$, where π_i denotes the i-th projection. Assume that there exists a map $\xi : F \to \{\pi_1, \pi_2, \pi_3\}$ which preserves composition. Is there a continuous such map?

Problem 2: Same question as above, assuming that the set of functions $\{x \mapsto f(x, x, x) : f \in F\} \subset \omega^\omega$ contains an oligomorphic group.

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2.9 Maurice Pouzet, Université Claude-Bernard / University of Calgary

First, here are two old problems about the notions of indivisibility and minimality introduced by Roland Fraïssé. A relational structure R is *indivisible* (resp. *age-indivisible*) if for every partition of its domain into two parts, say A, B, one of the induced structures $R_{\restriction A}$ and $R_{\restriction B}$ embeds R (resp. has the same age as R). See [2]. An indivisible structure is obviously age-indivisible.

Problem 1: If R is age-indivisible, does there is an indivisible relation with the same age as R? Note that the answer is yes if the maximum of the arities is 3. See [5].

A relational structure R is *minimal for its age* if every induced substructure with the same age as R embeds R (see [2]). Not every age has a structure which is minimal for this age (see [4]), but, trivially, if R is age-indivisible and minimal for its age, then R is indivisible.

Problem 2: If R is age-indivisible, is there R' with the same age that is also minimal for its age?

Next, here are some problems on the collection of orbits of an oligomorphic group. The *profile* of a relational structure M is the function φ_M which counts for every non-negative integer n the number $\varphi_M(n)$ of n-element substructures of M counted up to isomorphy, see [6]. When M is homogeneous, this counts the orbits of the action of $Aut(M)$ on the n-element subsets of the domain of M. The *age of a group G of permutations on a set* V is the set $\mathrm{Age}(G)$ of orbits of finite subsets of V. This set can be ordered as follows:

for two orbits O', O'' we set $O' \leq O''$ if there are subsets F' and F'' of V such that $F' \subseteq F''$, $O' = Orb(F')$ and $O'' = Orb(F'')$. As an ordered set, $\mathrm{Age}(G)$ is ranked; elements of rank n being the orbits of n-element subsets of V. The function φ_G which counts for each integer n the (cardinal) number $\varphi_G(n)$ of orbits of n-element subsets is the the *orbital profile* of G. If the number of these orbits is finite for each integer n then G is *oligomorphic* (see [1]).

Problem 3: If φ_G is not bounded above by some exponential function of n then $\mathrm{Age}(G)$ contains an infinite antichain.

The following result yields groups whose ages have no infinite antichain (use the test given in [7]).

Theorem Let M be a relational structure. If for every non-negative integer n the class $\mathrm{Age}_{n^-}(M)$ of finite substructures N of M with n-labelled elements, say $(N, \{a_1, \dots a_n\})$, has no infinite antichain, then there is some M' with the same age as M whose theory is $\aleph_0$-categorical and inductive. Moreover, if the set of initial segment of $\mathrm{Age}_{\omega^-}(M) := \bigcup_{n<\omega} \mathrm{Age}_{n^-}(M)$ has no infinite antichain then $\mathrm{Age}(G')$, where $G' := Aut(M')$, has no infinite antichain.

Finally, here are some problems about polynomially bounded profiles. Cameron conjectured that if G is a permutation group, φ_G is bounded above by some polynomial function of n then $\varphi_G(n) \simeq a.n^k$ for some $a > 0$ and $k \in \mathbb{N}$ (see [1]). Macpherson [3] asked if the fact that φ_G is bounded above by some polynomial function implies that the Cameron algebra of G is finitely generated (this algebra is made of linear combinations of members of $\mathrm{Age}(G)$ (see [1]). A positive answer implies that the generating series associated to the profile is a rational fraction; Cameron's conjecture would follow.

We may note that there are relational structures whose profile is bounded above by some polynomial and in fact the generating series is a rational fraction but for which the age algebra of Cameron is not finitely generated [8].

A description of those groups whose orbital profile is bounded above by some polynomial has yet to come. I propose the following approach: According to Schmerl [9] a relational structure M is *cellular* if its domain V is the disjoint union of a finite set F and a set which can be identified to the cardinal product $K \times L$ such that (1) for every permutation f of L the map $(1_K, f) \cup 1_F$ is an automorphism of M. I propose a variation of this notion, replacing (1) by the following condition:

(2) The substructures induced on two finite sets A and A' with the same cardinality are isomorphic provided that $(2a)$ $A \cap F = A' \cap F$ and $(2b)$ the frequency vectors $\chi_{A \setminus F}$ and $\chi_{A' \setminus F}$ are equal (the frequency vector $\chi_{A \setminus F}$ associates to every non-empty subset K' of K the number $\chi_{A \setminus F}(K') := |\{\ell \in L : A \cap (K \times \{\ell\}) = K' \times \{\ell\}\}$).

I will say that $M$is *set-cellular* if it has such a decomposition.

Problem 4:

1. If $\varphi(M)$ is bounded above by some polynomial then M is set-cellular (the converse holds trivially).
2. If M is set-cellular the generating series of the profile is a rational fraction.

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2.10 Norbert Sauer, University of Calgary

Problem: Let R be a relational structure on $\mathbb{N}$. Consider the poset $\mathbb{P}$ of all copies of R in itself, ordered by inclusion. How does it look like? For example, does it have an infinite antichain? Note that it is known that the cardinality of $\mathbb{P}$ is either 1 or continuum. Furthermore, if R is ω-categorical, the second alternative always holds and $\mathcal{P}(\mathbb{N})$ embeds into $\mathbb{P}$.

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2.11 John Truss, University of Leeds

In [2] it is shown that if p is a partial endomorphism of a finite (simple) graph Δ, then there is a finite graph $\Delta' \supseteq \Delta$ and a (totally defined) endomorphism of Δ' extending p.

Problem: Prove that if Δ is a finite graph, then there is a finite graph $\Delta' \supseteq \Delta$, such that every partial endomorphism of Δ extends to an endomorphism of Δ'.

This would be the analogue of Hrushovski's Lemma for partial automorphisms in the endomorphism context. Here, by "endomorphism" is understood a map such that for any two points of its domain, if they are joined by an edge in the graph, then so are their images (but two points which are not joined are allowed to be mapped to an edge, a non-edge, or collapsed to a point).

Note that there are many examples where the corresponding property is known to hold for partial automorphisms. In fact there is a large literature on it, see for example [1]. For partial endomorphisms, no instances are known (apart from the trivial structure).

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2.12 Andrew Zucker, Carnegie-Mellon University

Problem: Let G be a compactly approximable Polish group. Is G uniquely ergodic?

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3 Presentation Highlights and Scientific Progress Made

The main themes of the workshop lectures were categorized as follows.

- Structural Ramsey theory
- Continuous Fraïssé and topological groups
- Ramsey Properties of Homogeneous Structures
- Homogeneous structures, Morel Theory & Ramsey
- Clones, Reducts, and Constraint Satisfaction Problems

3.1 Structural Ramsey theory

Jaroslav Nešetřil (Charles University) opened the meeting with a broad overview on Structural Ramsey theory.

In a joint work with Dana Bartosova, Aleksandra Kwiatkowska (UCLA) followed with the topic in the projective setting. She presented some Ramsey theoretic statements about the dynamics of the homeomorphism group of the Lelek fan. One of the main results is a generalization of the finite Gowers' Ramsey theorem.

Micheal Pawliuk (University of Toronto) provided insight into the amenability and the Hrushovski property for Fraïssé classes of directed graphs (joint work separately with Miodrag Sokic and Marcin Sabok). Building on work of Angel-Kechis-Lyons and Zucker the amenability question has recently been answered for all Fraïssé classes of directed graphs, and using a Mackey-type construction the Hrushovski question has recently been investigated for the same classes. Micheal surveyed the results and the techniques used.

Jan Hubicka (University of Calgary) discussed Ramsey properties of multiamalgamation classes, in joint work with Nešetřil.

Andy Zucker (Carnegie Mellon University), discussed Ultrafilters and Structural Ramsey Theory, considering Ramsey objects and objects of finite Ramsey degree in a Fraïssé class K. He showed that an element of K is a Ramsey object if and only if a certain collection of ultrafilters is nonempty, providing a similar characterization of having finite Ramsey degree. These results apply to the dynamics of the automorphism group of the Fraïssé limit.

3.2 Continuous Fraïssé and topological groups

To start the day, Julien Melleray (University Lyon 1) presented results on Polish groups as automorphism groups of metric structures, giving a partial survey of what can be gained by approaching a Polish group as the automorphism group of a homogeneous metric structure. In particular, he used metric Fraïssé limits and the natural topometric structure on the automorphism group of a metric structure.

This was followed by Martino Lupini (California Institute of Technology) on the Poulsen simplex and Fraïssé theory for metric structures. The Poulsen simplex is the unique metrizable Choquet simplex with dense extreme boundary. He explained how one can study the Poulsen simplex from the perspective of Fraïssé theory for metric structures, and how many classical results about the Poulsen simplex can be recovered in this framework. He concluded by mentioning how this point of view allows one to define and construct the noncommutative analog of the Poulsen simplex.

Todor Tsankov (Université Paris 7) presented his joint work with Itaï Ben Yaacov and Tomás Ibarlucía on the Banach representations of dynamical systems and model theory. It is well-known that the automorphism group of an omega-categorical structure encodes all model-theoretic information about the structure. But recently an interesting correspondence has been discovered between properties of the theory (stability, omega-stability, NIP) and classes of Banach spaces on which certain dynamical systems (the automorphism group acting on type spaces over the model) can be represented. In the stable case, those dynamical systems also carry the structure of a semigroup that can be exploited. He discussed what is known about this correspondence as well as some open questions.

Slawomir Solecki (University of Illinois at Urbana-Champaign) discussed Fraïssé limits and topological spaces. He remarked that the pseudoarc is a remarkable compact connected space, in fact, it is the generic compact connected space. He explained the connection between the pseudoarc and projective Fraïssé limits coming for joint work with Trevor Irwin: the pseudoarc is represented as a quotient of such a limit. Further, he described recent work with Todor Tsankov, in which they determined the correct partial homogeneity of the projective Fraïssé limit associated with the pseudoarc. This determination involves combinatorial and basic ”dual” model theoretic arguments (e.g., a notion of dual type). He also described a transfer theorem, through which he recovers Bing's homogeneity of the pseudoarc from our partial homogeneity of the projective Fraïssé limit. He cncluded with recent work with Aristotelis Panagiotopoulos on the Menger curve viewed as a quotient of another projective Fraïssé limit.

Dragan Masulovic (University of Novi Sad) discussed the Kechris-Pestov-Todorcevic correspondence for projective Fraïssé limits. He presented a way to reinterpret the Kechris-Pestov-Todorcevic correspondence in an abstract categorical setting, to then instantiate this abstract setting in several ways. The interpretation in the category of countable structures with embeddings gives the well-known results of K-P-T theory for Fraïssé limits. The interpretation of this setting in categories of arbitrary structures with embeddings yields some recent results of Bartošova in which extreme amenability of automorphism groups of some uncountable structures was established. Finally, the interpretation in op-categories yields duals of some results of K-P-T theory. For example, he showed that if F is a projectively homogeneous structure, then Aut(F) is extremely amenable if and only if the projective age of F has the dual Ramsey property.

Wieslaw Kubis (Academy of Sciences of the Czech Republic) described category-theoretic framework for Fraïssé limits, capturing objects outside of model theory. The basic setting here is a category enriched over metric spaces plus a function measuring the “distortion” of arrows. Within this scheme, and adding some natural axioms, the Fraïssé limit exists, is unique, and has similar properties to classical Fraïssé limits. His approach is parallel to Ben Yaacov's continuous Fraïssé theory, but trying to avoid model-theoretic issues. Within this framework, he captures the Gurarii space, the pseudo-arc, the Poulsen simplex, and some other objects coming from analysis and topology (both new and existing ones).

3.3 More on Ramsey Properties of Homogeneous Structures

Norbert Sauer (University of Calgary) started the day discussing partitions of automorphism groups, in particular since in general partition properties of homogeneous structures are properties of their automorphism group. He argued that as it is difficult to characterize homogeneous structures it might be better to try to obtain partition results for subgroups of the symmetric group directly instead of for homogeneous structures. Notions and results and unresolved issues arising in this context were discussed.

Maurice Pouzet (University Lyon 1) presented an overview of the Equimorphy versus Isomorphy problem, in joint work with Claude Laflamme, Norbert Sauer and Robert Woodrow. Here two structures are said to be equimorphic if each embeds into the other. Maurice reported on two conjectures about the number of structures (counted up to isomorphy) which are equimorphic to a given structure; one by Bonato and Tardif asking whether the number of trees equimorphic of a given tree is either 1 or is infinite, the other by Thomassé asking a similar question for relational structures. He presented a positive answer of Thomassé's conjecture for chains and for countable homogeneous structures (whose automorphism group is oligomorphic). He concluded by some results about the hypergraph of copies of a countable homogeneous structure.

3.4 Homogeneous structures, Model Theory & Ramsey

David Evans (University of East Anglia) discussed his contributions to Topological dynamics of automorphism groups of Hrushovski constructions. Indeed using Hrushovski's predimension construction, he showed that there exists a countable, omega-categorical structure M with the property that if H is an extremely amenable subgroup of the automorphism group of M, then H has infinitely many orbits on its square. In particular, H is not oligomorphic. This answers a question raised by several authors (including Bodirsky, Pinsker, Tsankov and Nešetřil). It follows that there is a closed, oligomorphic permutation group G whose universal minimal flow M(G) is not metrizable.

David Bradley-Williams (University of Central Lancashire) discussed reducts of primitive Jordan structures, structures which have an automorphism group which is a primitive Jordan group. This means that the automorphism group acting on M is a Jordan group which preserves no non-trivial, proper equivalence relations on M. David first gave a brief survey of examples where results on Jordan groups have been used to obtain results about reducts of primitive Jordan structures. In particular, he discussed the classification of reducts, up to interdefinability, of any relatively 2-transitive semilinear ordering.

Robert Gray (University of East Anglia) discussed set-homogeneous structures. Here a countable relational structure M is called set-homogeneous if whenever two finite substructures U, V of M are isomorphic, there is an automorphism of M taking U to V (but not requiring that every isomorphism between U and V extends to an automorphism). This notion was originally introduced by Fraïssé, although unpublished observations had been made on it earlier by Fraïssé and Pouzet. Clearly every homogeneous structure is set-homogeneous. It is also not too difficult to construct examples of structures that are set-homogeneous but not homogeneous. It is natural to investigate the extent to which homogeneity is stronger than set-homogeneity, and this question has received some attention in the literature. For instance, it was shown by Ronse that any finite set-homogeneous graph is in fact homogeneous. In this talk Robert gave a survey of some of the known results in this area, including results on countably infinite set-homogeneous graphs due to Droste, Giraudet, Macpherson and Sauer, and results on set-homogeneous directed graphs obtained in recent joint work with Macpherson, Praeger and Royle. He concluded with a number of interesting conjectures and open problems that remain about set-homogeneous structures.

John Truss (University of Leeds) discussed countable homogeneous lattices in joint work with Aisha Abogatma. Previously a rather short list of countable homogeneous lattices was known, including, apart from the one-point lattice and the rationals, the countable universal-homogeneous distributive lattice and one or two others arising from amalgamations of certain classes of lattices. They showed that there are in fact uncountably many countable homogeneous lattices. Their examples are all non-modular, and the natural question to ask is whether every countable homogeneous modular lattice is necessarily distributive, a conjecture which has recently been proved by Christian Herrmann. Their method also applies to show that certain other classes of structures also have uncountably many countable homogeneous members.

Gabriel Conant (University of Illinois at Chicago) provided an overview of model theory of generalized Urysohn spaces. He argued that many well known examples of homogeneous metric spaces and graphs can be viewed as analogs of the rational Urysohn space (for example, the random graph as the Urysohn space with distances 0,1,2). In 2007, Delhomme, Laflamme, Pouzet, and Sauer characterized the countable subsets S of nonnegative reals for which an "S-Urysohn space" exists. Sauer later showed that, under mild closure assumptions on S, the existence of the S-Urysohn space is equivalent to associativity of a natural binary operation on S induced by usual addition of real numbers. In this talk, Gabriel considered the R-Urysohn space, where R is an arbitrary ordered commutative monoid. He first constructed an extension R* of R, such that any model of the theory of the R-Urysohn space (in a discrete relational language) can be given the structure of an R*-metric space. He then characterized quantifier elimination in this theory by continuity of addition in R*. Finally, he also characterized various model theoretic properties of the R-Urysohn space using natural algebraic properties of R.

Caroline Terry (University of Illinois at Chicago), presented an application of model theoretic Ramsey theory. As Chudnovsky, Kim, Oum, and Seymour recently established that any prime graph contains one of a short list of induced prime subparts, Caroline presented joint work with Malliaris in which they reprove their theorem using many of the same ideas, but with the key model theoretic ingredient of first determining the so-called amount of stability of the graph. This approach changes the applicable Ramsey theorem, improves the bounds, and offers a different structural perspective on the graphs in question.

Matthias Hamann (University of Hamburg) discussed Connected-homogeneous digraphs, which is a directed graph where any isomorphism between every two finite connected subdigraphs extends to an automorphism of the digraph. Matthias discussed the classification of the countable such digraphs, including a description of the main classes of these digraphs as well as a discussion of the main steps in the proof of the classification. In the end he provided arguments showing that their classification is on the one hand complete but on the other hand still incomplete.

David S. Gunderson (University of Manitoba) concluded the day with a discussion on Ramsey arrows for graphs. A simple form of Ramsey's theorem says that for any positive integer m, there exists an n=R(m) so that no matter how the pairs of an n-set are partitioned into two colours, some m-subset has all its pairs the same colour. In terms of graphs, this says if the edges of a Kn are 2-coloured, a monochromatic copy of Km (as a subgraph) can always be found. Such a statement is often expressed in "Ramsey arrow" notation. He then provided a short survey of Ramsey arrows for graphs, culminating in a characterization found with Rodl and Sauer of those triples G,H,r for which there is an F that arrows G when colouring Hs with r colours.

3.5 Clones, Reducts, and Constraint Satisfaction Problems

The final day of the workshop started with Manuel Bodirsky (TU Dresden) on applications of homogeneous structures in computer science. Homogeneous structures and their reducts have been used as templates of Constraint Satisfaction Problems (CSPs) to model qualitative reasoning problems in Artificial Intelligence. But this is not the only context in which homogeneous structures arise naturally in CS; Manuel discussed more recent links between homogeneous structures and permutation pattern avoidance classes, and between homogeneous structures and automata theory (for data word languages). He further presented a fragment of existential second-order logic such that the queries that can be formulated in this logic describe (finite unions of) CSPs for reducts of homogeneous structures. This logic is quite powerful and contains MMSNP and most CSPs that have been studied in temporal and spatial reasoning.

Michael Kompatscher (Vienna University of Technology) provided a counterexample on the reconstruction of oligomorphic clones. Two omega-categorical structures are first order bi-interpretable iff their automorphism groups are isomorphic as topological groups. For many well-known omega-categorical structures this statement still holds if we ignore the topology. But in 1990 Evans and Hewitt constructed two omega-categorical structures with isomorphic, but not topologically isomorphic automorphism groups. Similarly two omega-categorical structures are primitive positive bi-interpretable iff their polymorphism clones are topologically isomorphic. Based on the group-counterexample, Michael was able to construct a counterexample for the clones. This is a joint work with Manuel Bodirsky, David Evans and Michael Pinsker.

Michael Pinsker (Charles University Prague) concluded the workshop with a series of conjectures for clones over finitely bounded homogeneous structures. There has been a conjectured criterion, by Manuel Bodirsky and Michael, for when deciding the truth of a primitive positive sentence over a reduct of a finitely bounded homogeneous structure is tractable. This criterion has recently been replaced by a seemingly better criterion, although the equivalence of the two criteria is an open problem. Michael discussed the two conjectures, their relation, and further related conjectures and thoughts.

4 Outcome of the Meeting

The original goal and ultimately success of the meeting was to bring together the various groups working on the related area of homogeneous structures. In addition to tackling central problems, the workshop contained a substantial training component.

Finally this workshop can be viewed as the fourth international meeting on homogeneous structures, the first three being the following.

- London Mathematical Society Northern Regional Meeting and Workshop on Homogeneous Structures, July 19-22, 2011
- Workshop on Homogeneous Structures, Prague, July 25-27, 2012
- Workshop on Homogeneous Structures, Bonn, October 28-31, 2013

This was also an opportunity to celebrate Professor Sauer's 70th birthday and his contributions to the subject over 45 years.

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