Eigenvalues for Infinite Matrices, Their Computations and Applications

P.N.Shivakumar

Email: shivaku@cc.umanitoba.ca

(Collaborator: Yang Zhang)

Department of Mathematics

University of Manitoba,
Winnipeg, Canada R3T 2N2

UNIVERSITY OF MANITOBA

Abstract

In this talk, we discuss

- properties of eigenvalues which occur in infinite linear algebraic systems of the form $Ax = \lambda x$, where A is an infinite diagonally dominant matrix and b is a bounded linear operator,
- the classical question "Can you hear the shape of a drum?": "YES" for certain convex planar regions with analytic boundaries; "NO" for some polygons with reentrant corners,
- eigenvalues and van der Waerden infinite matrix.

History

Earliest concept of infinity comes from Anaximander, a Greek and consequently mathematical infinity is attributed to ZENO (400 B.C.). Also. The Indian text Surya Prajnapti (300-400 B.C.) classified all numbers into three sets, namely, enumerable, innumerable, and infinite. In 1655, Waller introduced the symbol ∞.

"Two things are infinite: the universe and human stupidity; I am not sure about the universe." ——— A. Einstein

Discussion of infinite systems generally start with truncated finite systems.

References:

(a) "A History of Infinite Matrices" by Michael Bernkopf

(b) "Infinite Matrices and Sequence Space" by R. G. Cooke

Some earlier authors are Poincare (1984, Hill's equation), Hilbert (1906, Hilbert infinite quadratic forms, equivalent to infinite matrices and Fredholm integral equations), John von Neumann (1929, Infinite matrices as operators) and Richard Bellman (1947, The boundedness of solutions of infinite systems of linear equations).

We give three different criteria for $A_{n\times n} = (a_{ij})$ to be nonsingular.

(a) Diagonal dominance

$$\sigma_i |a_{ii}| = \sum_{j=1,\ j\neq i}^{n} |a_{ij}|,\ 0 \leq \sigma_i < 1,\ i = 1, 2, ..., n.$$

(b) A chain condition

For $A = (a_{ij})_{n\times n}$, there exists, for each $j \notin J$, a sequence of nonzero elements of the form $a_{i,i_1}, a_{i,i_2}, ..., a_{i,i_j}$ with $j \in J$.

(c) Some sign patterns

$$\begin{cases} a_{ij} > 0 \quad \text{for } i \leq j, \\ (-1)^{i+j} a_{ij} > 0 \quad \text{for } i > j, \end{cases} \qquad (i, j = 1, 2, ..., n)$$

Linear Algebraic Systems

Consider

$$\sum_{j=1}^{\infty} a_{ij} x_j = b_i,\ \ i = 1, 2, ..., \infty,$$

where

- the infinite matrix $A = (a_{ij})$,
- $\sigma_i |a_{ii}| = \sum_{j=1}^{\infty} |a_{ij}|;\ \ 0 \leq \sigma_i < 1,\ \ i = 1, 2, ..., \infty,$
- the sequence $\{b_i\}$ is bounded.

We establish sufficient conditions which guarantee the existence and uniqueness of a bounded solution to the above system as well as bounds for truncated systems.

We have

$$|x_j^{(n+1)} - x_j^{(n)}| \leq P\sigma_{n+1} + Q|a_{n+1,n+1}|^{-1}$$

for some positive constraints P and Q.

An estimate for the solution is given by

$$|\underline{x}^{(n)}| \leq \prod_{k=1}^{n} \frac{1+\sigma_k}{1-\sigma_k} \sum_{k=1}^{n} \frac{|b_k|}{|a_{kk}|(1+\sigma_k)}.$$

Linear Eigenvalue Problem

To determine the location of eigenvalues for diagonally dominant infinite matrices (which occur in (a) solutions of elliptic partial differential equations and (b) solutions of second order linear differential equations) and establishes upper and lower bounds of eigenvalues, we define an eigenvalue of A to be any scalar λ for which $A\underline{x} = \lambda\underline{x}$ for some $0 \neq \underline{x} \in D(A)$.

Assume A is an operator on l_1.

$E(1)$: $a_{ii} \neq 0, \; i \in \mathbb{N}$ and $|a_{ii}| \to \infty$ as $i \to \infty$.

$E(2)$: $\exists \rho \in [0, 1)$, s.t. for each $j \in \mathbb{N}$

$$Q_j = \sum_{i=1, i \neq j}^{\infty} |a_{ij}| = \rho_j |a_{jj}|, \; 0 \leq \rho_j \leq \rho.$$

$E(3)$: For all $i, j \in \mathbb{N}$, $i \neq j$, $|a_{ii} - a_{jj}| \geq Q_i + Q_j$.

$E(4)$: For all $i \in \mathbb{N}$, $\sup\{|a_{ij}| : \; j \in \mathbb{N}\} < \infty$.

Linear Differential System

We consider the infinite system

$$\frac{d}{dt}x_i(t) = \sum_{j=1}^{\infty} a_{ij}x_j(t) + f_i(t),\ \ t \geq 0,\ \ x_i(0) = y_i,\ \ i = 1, 2, ...$$

In particular, if A is a bounded operator on l_1, then the convergence of a truncated system has been established.

(Arley and Brochsenius (1944), Bellman (1947) and Shaw (1973) have studied this problem. However, none of these papers yields explicit error bounds for such a truncation.)

Applications

1. Digital circuit dynamics
2. Conformal mapping of doubly connected regions
3. Fluid flow in pipes
4. Mathieu equation $\frac{d^2y}{dx^2} + (\lambda - 2q\cos 2x)y = 0$ for a given q

   with the boundary conditions: $y(0) = y(\frac{\pi}{2}) = 0$

   (Our interest: two consecutive eigenvalues merge and become equal for some values of the parameter q.)

5. Bessel Function
6. Eigenvalues of the Laplacian on an elliptic domain

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \lambda^2 u = 0 \text{ in } D, \;\; u = 0 \text{ on } \partial D,$$

where D is a plane region bounded by smooth curve ∂D.

Consider the domain bounded by the ellipse represented by $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1$, which can be expressed correspondingly in the complex plane by

$$(z + \overline{z})^2 = a + bz\overline{z},$$

where $a = \frac{4\alpha^2\beta^2}{\beta^2 - \alpha^2}, \;\; b = \frac{4\alpha^2}{\alpha^2 - \beta^2}$.

After considerable manipulation, we get the value of u on the ellipse as

$$
\begin{aligned}
u = 2a_0 &+ \sum_{n=1}^{\infty} A_{2n,0} B_{0,n} a_n + \sum_{k=1}^{\infty} \left(-\frac{\lambda^2}{4}\right)^k \frac{2a_0}{k!k!} (z\bar{z})^k \\
&+ \sum_{k=1}^{\infty} \left[\sum_{n=1}^{k} \left(A_{2n,k} b_{0,n} + \sum_{l=1}^{n} A_{2n,k-l} b_{l,n} \right) a_n \right] (z\bar{z})^k \\
&+ \sum_{k=1}^{\infty} \left[\sum_{n=k+1}^{\infty} \left(A_{2n,k} b_{0,n} + \sum_{l=1}^{k} A_{2n,k-l} b_{l,n} \right) a_n \right] (z\bar{z})^k .
\end{aligned}
$$

For $u = 0$ on the elliptic boundary, we equate the powers of $z\bar{z}$ to zero, where we arrive at an infinite system of linear equations of the form

$$\sum_{k=0}^{\infty} d_{kn} a_n = 0, \;\; n = 0, 1, 2, ..., \infty,$$

where d_{kn}'s are known polynomials of λ^2. The infinite system is truncated to $n \times n$ system and numerical values were calculated and compared to existing results in literature. The method derived here provides a procedure to numerically calculate the eigenvalues.

8. Mathieu Equation

Shape of A Drum

For the classical question, “Can you hear the shape of a drum?”, the answer is known to be “yes” for certain convex planar regions with analytic boundaries. The answer is also known to be “no” for some polygons with reentrant corners. A large number of mathematicians over four decades have contributed to the topic from various approaches, theoretical and numerical.

We develop a constructive analytic approach to indicate how a preknowledge of the eigenvalues leads to the determination of the parameters of the boundary. This approach is applied to a general boundary and in particular to a circle, an ellipse, an annulus and a square. In the case of a square, we obtain an insight into why the analytical procedure does not, as expected, yield an answer.

The pursuit of a "complete" solution to the question "Can one hear the shape of a drum", originally posed by Lipman Bers and used as a title in 1966 by Mark Kac, has been a fascinating journey and work can only be described as "work in progress". The research has involved many mathematicians and many tools including Asymptotics, Probability Theory, Operator Theory, infinite algebraic systems and inevitably intense computational work involving approximate methods.

Mathematically, the problem is, whether a preknowledge of the eigenvalues of the Laplacian in a region Ω leads to the definition of Γ, the closed boundary of Ω. Specifically, we have

$$u_{xx} + u_{yy} + \lambda^2 u = 0 \text{ in } \Omega, \tag{1}$$

$$u = 0 \text{ on } \Gamma. \tag{2}$$

According to the maximum principle for linear elliptic partial differential equations, the infinite eigenvalues $\lambda_n^2,\ n = 1, 2, 3, ...,$ are positive, real, ordered and satisfy

$$0 < \lambda_1^2 < \lambda_2^2 < \cdots < \lambda_n^2 < \cdots < \infty. \tag{3}$$

In the famous paper "Can you hear the shape of a drum?", Mark Kac makes an analysis using only asymptotic properties of large eigenvalues and uses probability theory as the tool to establish that one can hear the area of a polygonal drum and conjectures for multiply connected regions.

A major contribution in 2000 is given by Zelditch, where a positive answer "yes" is given for certain regions with analytic boundaries.

Preliminaries

We express (1) in terms of complex variables $z = x + iy$, $\bar{z} = x - iy$, and the system (1) and (2) becomes

$$u_{z\bar{z}} + \frac{\lambda^2}{4} u = 0 \text{ in } \Omega, \tag{4}$$

$$u = 0 \text{ on } \Gamma. \tag{5}$$

We start the analysis using the completely integrated form of the solution to (1) as given in Vekua by

$$u = \left\{ f_0(z) - \int_0^z f_0(t) \frac{\partial}{\partial t} J_0 \left(\lambda \sqrt{\overline{z}(z-t)} \right) dt \right\} + \text{ conjugate}, \quad (6)$$

where $f_0(z)$ is an arbitrary analytic function which can be formally expressed as

$$f_0(z) = \sum_{n=0}^{\infty} a_n z^n \qquad (7)$$

and J_0 represents the Bessel function of first kind and order 0 given by

$$J_0 \left(\lambda \sqrt{\overline{z}(z-t)} \right) = \sum_{q=0}^{\infty} \left(-\frac{\lambda^2}{4} \right) \frac{\overline{z}^q (z-t)^q}{q! q!}.$$

From an identity given in Abramowitz and Stegun, we have, when n is even,

$$z^n + \overline{z}^n = \sum_{m=0}^{\frac{n}{2}} c_{mn}(z + \overline{z})^{n-2m}(z\overline{z})^m, \;\; n = 2, 4, 6, ...,$$

Boundary Γ

(a) **A general boundary**
We consider the parametrized analytical boundary Γ with biaxial symmetry to be given by

$$(z+\bar{z})^{2n} = \sum_{n=0}^{\infty} d_{n1}(z\bar{z})^n$$

which yields, on using Cauchy products for infinite series,

$$(z+\bar{z})^{2n}p = \sum_{n=0}^{\infty} d_{np}(z\bar{z})^n,$$

where

$$d_{np} = \sum_{l=0}^{n} d_{l\ p-1} d_{n-l\ 1},\ \ p = 1,2,3,\ldots.$$

(b) **Circular boundary**

We consider the circular boundary given by

$$x^2 + y^2 = a^2 \text{ or } z\bar{z} = a^2.$$

(c) **An elliptic boundary**

We consider the elliptic boundary Γ given by

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1 \text{ or } (z + \bar{z})^2 = a + bz\bar{z}, \ \alpha > 0.$$

(d) **A square boundary**

We give this example to demonstrate why our analytical approach does not yield information of the boundary with sharp corners from a preknowledge of eigenvalues. Consider a square boundary given by

$$x = \pm a,\ \ y = \pm a \ \text{ or } \ z^4 + \bar{z}^4 = 2(z\bar{z})^2 - 16a^2(z\bar{z}) + 16a^4$$

$$\text{or } z^2 + \bar{z}^2 = 4(z\bar{z} - 2a)^2.$$

We assume

$$f_0(z) = \sum_{n=0}^{\infty} a_{2n} z^{2n},$$

$$u = 2a_0 J_0\left(\lambda\sqrt{z\overline{z}}\right) + \sum_{n=1}^{\infty} a_{2n} \sum_{k=0}^{\infty} \left(-\frac{\lambda^2}{4}\right) A_{2n\ k}(z^{2n} + \overline{z}^{2n})(z\overline{z})^k,$$

which on substitution gives

$$u = 2a_0 J_0(\lambda\sqrt{z\overline{z}}) +$$

$$\sum_{n=1}^{\infty} a_{2n} \sum_{k=0}^{\infty} \left(-\frac{\lambda^2}{4}\right) A_{2n\ k} \sum_{m=0}^{n} (-1)^m \frac{2n(2n-m-1)!}{m!(2n-2m)!} (z+\overline{z})^{2(n-m)} (z\overline{z})^m.$$

(a) General boundary Γ

After rearrangement of summations, we get

$$
\begin{aligned}
u &= 2a_0 \\
&+ \sum_{n=1}^{\infty} a_{2n} D_{n\ 0\ 0\ 0} + \left[2a_0 \left(-\frac{\lambda^2}{4} \right) + \sum_{n=1}^{\infty} a_{2n} \left\{ \sum_{i=0}^{1} \sum_{p=0}^{1-i} D_{n\ p\ 1-i-p\ i} \right\} \right] z\overline{z} \\
&+ \sum_{q=2}^{\infty} \left\{ 2a_0 \left(-\frac{\lambda^2}{4} \right)^q \frac{1}{q!q!} + \sum_{n=1}^{q-1} a_{2n} \left[\sum_{i=0}^{n} \sum_{p=0}^{q-1} D_{n\ p\ q-i-p\ i} \right] \right. \\
&+ \left. \sum^{\infty} a_{2n} \left[\sum^{q} \sum^{q-1} D_{n\ p\ q-i-p\ i} \right] \right\} (z\overline{z})^q.
\end{aligned}
$$

Now we equate the coefficients of $(z\bar{z})^q$, $q = 0, 1, 2, \ldots$ to zero to obtain the infinite linear algebraic system. Writing the resulting system as

$$\sum_{n=0}^{\infty} f_{qn} a_{2n}, \;\; q = 0, 1, 2, \ldots,$$

we note that f_{qn} is a polynomial of degree q in μ and contains only d_{01}, d_{11}, d_{21},.... The values of μ are determined by formally setting $D = \det(f_{qn}) = 0$, which can be expressed as an infinite series in μ given by

$$D = \sum_{i=0}^{\infty} D_i \mu^i$$

and μ_i's are given by $D = 0$.

We note that we can write

$$D = D_0 \prod_{i=1}^{\infty} \left(1 - \frac{\mu}{\mu_i}\right)$$

formally suggesting that $\frac{D_i}{D_0}$ is the sum of the inverse products of μ_1, μ_2,... taken i at a time. In particular,

$$\frac{D_1}{D_0} = -\sum_{i=1}^{\infty} \frac{1}{\mu_i}, \qquad \frac{D_2}{D_0} = \sum_{i=1}^{\infty} \sum_{j=i+1}^{\infty} \frac{1}{\mu_i \mu_j}$$

(b) A circle

It is well known that the solutions to (1) and (2) is given by

$$u = a_0 J_0\left(\lambda\sqrt{z\overline{z}}\right)$$

and the eigenvalues are given by the zeros of $J_0(\lambda a) = 0$, namely,

$$\lambda_i = \frac{j_{0i}}{a}, \; i = 1, 2, \ldots .$$

We can write

$$\sum_{q=0}^{\infty}\left(-\frac{\lambda^2}{4}\right)^q \frac{1}{q!q!} a^{2q} = \prod_{j=1}^{\infty}\left(1 - \frac{\lambda^2}{\lambda_j^2}\right).$$

We obtain

$$a^2 = 4 \sum_{j=1}^{\infty} \frac{1}{\lambda_j^2},$$

thus establishing uniquely the value of a if all the eigenvalues λ_1, λ_2,... are known. In fact, substitute for λ_i from $\lambda_i = \frac{j_{0i}}{a}$ we obtain

$$\frac{a^2}{4} = \sum_{j=1}^{\infty} \frac{a^2}{j_{0\ j}^2}$$

which checks with the well-known fact that

$$\sum_{j=1}^{\infty} \frac{1}{j_{0\ j}^2} = 4.$$

(c) An ellipse

Here we develop the solution in the elliptic domain. We get

$$\sum_{n=0}^{\infty} f_{nq} a_{2n} = 0, \;\; q = 0, 1, 2, \ldots.$$

We note that f_{nq} is a polynomial of degree q in μ. The determinant D_1 is the sum of an infinite determinants derived from differentiating D with respect to μ and setting $\mu = 0$.

(d) A square boundary

We give this example to demonstrate how our approach does not yield information of the boundary from a preknowledge of the eigenvalues in the case of a boundary containing sharp corners. Consider the square boundary

$$z^2 + \bar{z}^2 = 4(z\bar{z} - 2a)^2.$$

We use $f_0(z) = \sum_{n=0}^{\infty} a_{4n} z^{4n}$ to obtain

$$u = 2a_0 J_0\left(\lambda\sqrt{z\bar{z}}\right) + \sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\left(-\frac{\lambda^2}{4}\right)^k A_{4n\ k}(z^{4n} + \bar{z}^{4n}) a_{4n}(z\bar{z})^k,$$

$$z^{4n} + \bar{z}^{4n} = \sum_{m=0}^{n} b_{mn}(z^2 + \bar{z}^2)^{2n-2m}(z\bar{z})^{2m},$$

which gives $z^{4n} + \bar{z}^{4n} = \sum_{m=0}^{n} b_{mn} 4^{n-m}(z\bar{z} - 2a)^{n-m}$.

It is well-known that for a square, the eigenvalues λ^2 are given by $\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{a^2}$ and using $\mu = -\frac{\lambda^2}{4}$ yields

$$\frac{D_1}{D_0} = -\sum_{i=1}^{\infty} \frac{1}{\mu_i} = \frac{4a^2}{\pi^2} \sum_{m=1}^{\infty} \sum_{n=m+1}^{\infty} \frac{1}{m^2 + n^2}.$$

The double series is clearly divergent and hence the failure of the analytic approach, although $\det[g_{qn}] = 0$ yields the eigenvalues.

Conclusions

In this article, we have demonstrated a constructive approach when the answer is "yes". We give an interpretation to "a preknowledge of eigenvalues" as knowing the sums of the inverse products of the eigenvalues $\mu_i,\ i = 1, 2, ...$, taken i at a time. In the case of an ellipse, we express the parameters of the ellipse in terms of the eigenvalues. Our conjecture is that for the answer to be "yes", it is necessary for the sums of the inverse products of the eigenvalues $\mu_i,\ i = 1, 2, ...$, taken i at a time to be convergent series. Another conjecture is that all analytic curves (curvilinear polygons for example) yield the answer "yes".

Hearing the Shape of a Drum with a Hole

We are interested in the Helmholtz equation

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \lambda u = 0 \text{ in } \Omega$$

where Ω is the doubly connected region with the boundary conditions

$$u = 0 \text{ on } \pi_1 : \; x^2 + y^2 = a^2$$
$$u = 0 \text{ on } \pi_2 : \; x^2 + y^2 = b^2 \qquad a > b$$

We need to show that if the eigenvalues of λ are known, the ratio of the annulus is uniquely determined.

Equation $\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \lambda u = 0$ in polar coordinates (r, θ) and noting that $u = u(r)$, we get the Bessel equation

$$\frac{d^2u}{du^2} + \frac{du}{dr} + \lambda u = 0$$

whose solution is given in Ω [Garabedian].

$$U = c_1 J_0(\sqrt{\lambda} r) + c_2 \left\{ J_0(\sqrt{\lambda} r) \ln(\sqrt{\lambda} r) - \sum_{n=1}^{\infty} a_n H_n \lambda^n r^{2n} \right\}$$

where $a_n = \frac{(-1)^n}{2^n n!}$ and $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$.
Using the previous equations, we get

$$c_1 J_0(\sqrt{\lambda} a) + c_2 \left\{ J_0(\sqrt{\lambda} a) \ln(\sqrt{\lambda} a) - \sum_{n=1}^{\infty} a_n H_n \lambda^n a^{2n} \right\} = 0$$

$$c_1 J_0(\sqrt{\lambda} b) + c_2 \left\{ J_0(\sqrt{\lambda} b) \ln(\sqrt{\lambda} b) - \sum_{n=1}^{\infty} a_n H_n \lambda^n b^{2n} \right\} = 0$$

For non-zero c_1 and c_2, we need

$$J_0(\sqrt{\lambda}a)J_0(\sqrt{\lambda}b)\ln\left(\frac{b}{a}\right)+\left\{J_0(\sqrt{\lambda}b)\sum_{n=1}^{\infty}a_nH_n\lambda^n a^{2n}-J_0(\sqrt{\lambda}a)\sum_{n=1}^{\infty}a_nH_n\lambda^n b^{2n}\right\}=0$$

Equation above becomes an infinite series in λ. In fact, we can express the infinite series in the above equation by

$$\prod_{j=1}^{\infty}\left(1-\frac{\lambda}{\lambda_j}\right)\ln\left(\frac{b}{a}\right)$$

Equating the constants, we get $\ln\left(\frac{b}{a}\right) = \ln\left(\frac{b}{a}\right)$. And equating the coefficient of λ, we get

$$a_1(a^2+b^2)\ln\left(\frac{b}{a}\right) + a_1 H_1(a^2-b^2) = -\sum_{j=1}^{\infty}\frac{1}{\lambda_j}\ln\left(\frac{b}{a}\right).$$

Using $H_1 = 1$, $a = 1$ and $a_1 = -\frac{1}{2^4}$, we get

$$\frac{1}{2^4}(1+b^2) + \frac{1-b^2}{\ln(b)} = \sum\frac{1}{\lambda_j} = P.$$

Knowing P, we can numerically solve for b. We also note that conversely, if b is known, we get the value of

$$\sum_{j=1}^{\infty} \frac{1}{\lambda j}.$$

Hence, we can hear the shape of the drum which is an annulus if the eigenvalues are known.

Zeros of Infinite Series and $A\underline{x} = \underline{b}$

For the infinite series in λ, we will assume that the zeros of the series $\{\lambda_n\}$, $n = 1, 2, 3, \ldots$; we can write, with λ^2 replaced by x as

$$y(x) = \sum_{n=0}^{\infty} c_n x^{2n} = \prod_{n=1}^{\infty} (1 + a_n x)$$

where $c_0 = 1$, $c_1 = \sum_{k=1}^{\infty} a_k y'(0)$ and $\lambda_n = -\frac{1}{a_n}$.
We also have $0 < \lambda_1^2 < \lambda_2^2 < \lambda_3^2 < \cdots$. We seek to connect a's with c_n's.

The solution to

$$y'' + f(x)y = 0, \tag{8}$$

when $f(x)$ is an arbitrary function with continuous derivatives, is expressed by

$$\sum_{k=0}^{\infty} c_n x^n, \tag{9}$$

where c_n's are explicitly dependent on derivatives of $f(x)$ and the initial conditions on $y(x)$, namely, $y(0)$ and $y'(0)$.
In fact, $y(0) = c_0$ and $y'(0) = c_1$.

Reference: P. N. Shivakumar and Y. Zhang, On series solutions of $y''-f(x)y = 0$ and applications, Advances in difference equations, 2013, 201: 47.

It can be easily shown that, with

$$f(x) = \left(\sum_{n=1}^{\infty} \frac{a_n}{1 + a_n x} \right)^2 - \sum_{n=1}^{\infty} \frac{a_n^2}{(1 + a_n x)^2}, \tag{10}$$

solution to (8) is given by (9) in terms of Q_p.

We will use the notation $Q_p = \sum_{k=1}^{\infty} a_k^p, \; p = 1, 2, 3, \ldots$

Finally, we get

$$\sum_{k=1}^{\infty} a_k^p = e_p, \quad p = 1, 2, ..., \tag{11}$$

where e_p's are explicitly obtained as combinations of c_n's.

Rewriting the above equations as

$$\sum_{k=1}^{\infty} a_k^p x_k = e_p, \;\; p = 1, 2, ...,$$

we note that the system is of the form $A\underline{x} = \underline{e}$, where $\underline{x}$ is a unite vector and A is the well-known Vandermonde infinite matrix. We now get $\underline{x} = A^{-1}\underline{e}$. In all the discussion above, infinity can be replaced by a positive integer.

As a simple example, we consider a polynomial of degree 3 with increasing positive roots. By the above analysis, we arrive at the system

$$x_1 + x_2 + x_3 = c_1,$$
$$x_1^2 + x_2^2 + x_3^2 = c_2,$$
$$x_1^3 + x_2^3 + x_3^3 = c_3,$$

where c_1, c_2 and c_3 are determined by the coefficients of the polynomial. We arrive at the bound

$$0 < x_1 < \frac{c_2}{c_1} < x_2 < \frac{c_3}{c_2} < x_3 < \frac{c_1}{3}.$$

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Thank you!