A free boundary problem motivated by eigenfunctions localization

Guy David, Université de Paris-Sud

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Joint work with:
Marcel Filoche
Svitlana Mayboroda
David Jerison
Stephen Lewis (pictures)

Motivation

We are given an elliptic differential operator L on a bounded domain Ω. For instance,

$$L = -\Delta + \mathcal{V}$$

with a bounded nonnegative potential $\mathcal{V}$ (but reasonably wild, or take a complicated domain). Take Dirichlet boundary conditions on $\partial\Omega$.

Marcel and Svitlana: the localization of eigenfunctions is governed by the "Landscape function" u, solution of

$$Lu = 1 \text{ on } \Omega \text{ and } u = 0 \text{ on } \partial\Omega.$$

Attempt here: use u to define a functional J, whose minimizers give a decomposition of Ω into subdomains where eigenfunctions are likely to live.

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First attempt

General idea: try to decompose Ω into, say, N subdomains V_i, chosen so that u is small on the ∂V_i.
And to measure smallness, let us use a fake $H^{1/2}$-norm on $\Gamma = \cup_i \partial V_i$. To $L = -\Delta + \mathcal{V}$ we naturally associate an energy

$$E_L(w) = \int_\Omega |\nabla v|^2 + \mathcal{V} w^2$$

for $w \in H^1(\Omega)$, i.e., $w \in H^1(\mathbb{R}^n)$ such that $w = 0$ a.e. on $\mathbb{R}^n \setminus \Omega$.

First attempt: for each (nice) set Γ of boundaries, minimize $E_L(w)$ among the $w \in H^1(\Omega)$ such that $w = u$ on Γ. Then take Γ for which this is minimal.

Since $\Gamma = \emptyset$ minimizes, compensate by adding a volume term that prefers nontrivial partitions. For instance, the sum of the squares of the volumes of the components of $\Omega \setminus \Gamma$.

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Our functional

Finally, we prefer to work in terms $v = u - w$ and a collection of disjoint subsets $V_j \subset \Omega$, $1 \leq j \leq N$ (some of the components of $\Omega \setminus \Gamma$), and minimize the following functional:

$$J(v, \underline{V}) = F(V_1, \ldots, V_N) + \sum_{j=1}^{N} \int_{V_j} |\nabla v_j|^2 + v_j^2 \mathcal{V} - 2v_j$$

where we set $v_j = u \mathbb{1}_{V_j}$; thus $v_j \in H^1(\Omega)$ and $v_j = 0$ on $\Omega \setminus V_j$. This is essentially a transcription of the functional above; the term $\int_{V_j} v_j$ comes from the equation for u.

Let us just take

$$F(V_1, \ldots, V_N) = a \sum_{j=1}^{N} \mathrm{Volume}(V_j) + b \sum_{j=1}^{N} \mathrm{Volume}(V_j)^2$$

for some choice of $a, b > 0$. Explain about the black zone $\Omega \setminus \cup_j V_j$.

Comparizon with Alt-Caffarelli-Friedman

Recall:

$$J(\underline{v}, \underline{V}) = \sum_{j=1}^{N} \int_{V_j} |\nabla v_j|^2 + \sum_{j=1}^{N} \int_{V_j} (v_j^2 \mathcal{V} - 2v_j) + F(V_1, \ldots, V_N)$$
$$= E(\underline{v}) + M(\underline{v}) + F(\underline{V}) \qquad \text{(condensed notation)}$$

Here we may take N large (given in advance or autodetermined); Alt, Cafarelli, and Friedman's functional corresponds to taking $N = 1$ or $N = 2$, setting $v = v_1 - v_2$, and minimizing

$$J_{acf}(v) = \int |\nabla v|^2 + \int_{\{v>0\}} q_+ + \int_{\{v>0\}} q_-$$

for some positive functions q_+, q_-.

Main differences: we have a more general term $F(\underline{V})$, have a term $M(\underline{v})$ (not too serious I hope), and have N phases (the main issue).

Disclaimer/Apologies

From now on, not so much about the localization of eigenfunctions, which looks more like an excuse for studying a nice free boundary problem.

Much of what follows is quite similar to the original work of AFC, and many others later.
Hopefully new to some of you. The main difference is at the beginning (replace the harmonic extension arguments, when $N \geq 3$).

Independent work by Dorin Bucur and Bozhidar Velichkov, also on AFC functionals with more than two phases. They use a monotonicity formua with 3 phases.
Also see work of Ramos, Tavares, and Terracini.

The results on J

- Existence is not hard: $E(\underline{v})$ controls the other terms by Poincaré.

Main initial difficulty for the regularity of the minimizers: the usual harmonic competitor is not defined! In fact, we get as a corollary some control on harmonic functions with values in a spider.
Nonetheless, we follow the general plan for ACF:
- Minimizers are Hölder-countinuous (harder extra step here)
- They are Lipschitz (uses an important monotonicity formula from AFC)
- Some nondegeneracy properties. For instance, assuming $a, b > 0$,

$$v_j(x) \geq C^{-1} \operatorname{dist}(x, \mathbb{R}^n \setminus \{v_j > 0\})$$

- In any dimension, the free boundary $\partial_j = \partial\{v_j > 0\}$ is Ahlfors-regular and uniformly rectifiable (but more could be expected)

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Regularity 2

In dimension $n \leq 3$, maybe $n = 4$, we control all the blow-up limits; they are given by affine functions on half spaces, and the free boundaries are hyperplanes.
This uses earlier work of Caffarelli, Jerison and Kenig on "global minimizers".
- No point lies in (or very near) the boundary of three regions V_j. We are thus locally back in the situation with two phases!
- Under some additional nondegenacy conditions (to be discussed), we probably get the same regularity results as in the standard case (C^1 boundaries ∂_i in low dimensions or where ∂_i is flat).
[Long verification not done. But it is sad that the standard nondegenacy condition requires different volume forms in $F(\underline{V})$ for the different $i \leq N$?]

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Comments about what the functional does

We like to take $a, b > 0$ and allow a black zone (the V_j don't need to cover Ω).

Just because $a > 0$, we get the standard nondegeneracy condition.

And we can expect smooth free boundaries (except for the black zone).

$b > 0$ looks nice to give convexity and hopefully more equal volumes.

No problem with the boundary regularity. In fact the ∂V_j only approach $\partial\Omega$ tangentially (and the black zone fills the holes).

Autolimitation of N (with $a, b > 0$, we have a lower bound on the volume of each V_j (that does not depend on N), hence upper bound on the number of $V_j \neq \emptyset$.

But the theorem says nothing about the localization of eigenfunctions.

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Spiders

A spider with N legs is just the union S_N of N half lines L_j starting from 0. With the geodesic distance.

A harmonic function $\underline{v} : U \subset \mathbb{R}^n \to S_N$ is a local minimizer of

$$\int_U |\nabla \underline{v}|^2 = \sum_{j=1}^{N} \int_U |\nabla v_j|^2,$$

where v_j is the L_j-component of $\underline{v}$ (disjoint supports again).

What regularity for such a harmonic $\underline{v}$?
Notice: this is a free boundary problem!
In much more general spaces (Shoen, etc), $\underline{v}$ is Hölder. Here we get that $\underline{v}$ is Lipschitz (as a byproduct).
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A competitor

How do we replace the harmonic extension (without using harmonic functions on spiders)?

Or, what is a natural extension to a ball B of $\underline{v}_{|\partial B}$?
When $N = 2$, the positive and negative parts of the harmonic extension of $v_1 - v_2$ automatically have disjoint supports.

Often, when there is a dominant domain (say V_1) or function (v_1), proceed as follows:

Kill all the other functions v_j, $j > 1$, by a radial cut-off function on a small annulus. Keep v_1 as it is (extend radially).

Use the extra space on the smaller ball to extend v_1 harmonically.

Then, typically, compare this candidate to the minimizer.

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The ACF monotonicity formula

We are given two functions $v_1, v_2 \in H^1$, with $v_1 v_2 = 0$. Suppose also they are also Hölder, and that

$$\Delta v_i = 0 \ \text{ on } \{v_i > 0\}.$$

Then the product $F_1(r)F_2(r)$ is nondecreasing, where

$$F_i(r) = \frac{1}{r^2} \int_{B(0,r)} \frac{|\nabla v_i(x)|^2}{|x|^{n-2}} dx.$$

Idea: take a derivative, and use Poincaré's inequality on the two disjoint parts of the sphere where the v_i live. The optimal combination is when the disjoint parts are hemispheres, and the v_i are affine there. And then $F_1 F_2$ is constant.

Back to the localization of eigenfunctions

Pictures of computed minimizers: do they look like our theorem?

Picture of some eigenfunctions: do they live in the V_j? we take an extra risk, because of the black zones.

Thanks to Stephen Lewis (for the computations).