A sharp upper bound for the first Dirichlet eigenvalue of a class of wedge-like domains

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Setting

D will denote a planar domain given in polar coordinates by

$$D = \left\{ (r, \theta) \,|\, 0 < r < \rho(\theta), \;\; 0 < \theta < \frac{\pi}{\alpha} \right\}. \tag{1}$$

Then D is a bounded domain completely contained in the wedge $\mathcal{W}_\alpha$ of angle $\frac{\pi}{\alpha}$, $\alpha \geq 1$. We will denote by Γ the curved part of its boundary defined by $r = \rho(\theta), \;\; 0 \leq \theta < \frac{\pi}{\alpha}$. Let u be the fundamental eigenfunction of the Dirichlet problem in D, and λ its associated fundamental eigenvalue:

$$\Delta u + \lambda u = 0 \text{ in } D, \quad u = 0 \text{ on } \Gamma. \tag{2}$$

When Γ is convex (prototype for starlike) we know that $(x, n) > 0$ (Aissen 1958; Pólya-Szegö 1951, Siudeja-Laugesen 2014, Freitas-Krejčiřik 2008, etc.)

Setting

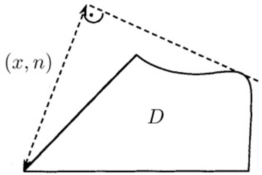
For such domains we define

$$B_\alpha = \int_\Gamma \frac{1}{(x,n)} h^2 d\sigma, \qquad A_\alpha = \int_D h^2 dA \tag{3}$$

Here, we let $\|x\| = r = \rho(\theta)$ and $h = r^\alpha \sin \alpha\theta$. We note that h is a harmonic nonnegative function in $\mathcal{W}_\alpha$. By a perfect sector, we mean the circular sector

$$S_0 = \left\{(r,\theta) \;\middle|\; 0 < r < R_0,\; 0 < \theta < \frac{\pi}{\alpha} \right\}$$

with R_0 the radius such that $|S_0| = \int_{S_0} h^2 dA = A_\alpha$.



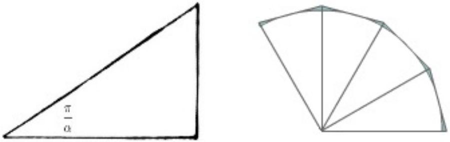
Main Theorem

The fundamental eigenvalue λ of the wedge-like membrane D defined above satisfies the (new) inequality in red.

$$\left(\frac{4\alpha(\alpha+1)}{\pi}A_\alpha\right)^{\frac{-1}{\alpha+1}} j_{\alpha,1}^2 \le \lambda \le \frac{B_\alpha}{(2\alpha+2)A_\alpha} j_{\alpha,1}^2. \tag{4}$$

Equality holds if and only if D is a perfect sector S_0.

Applications: (1) Estimates for λ, for the right-angle triangle with unit hypothenuse; (2) Estimates for λ for regular α-polygon with n sides.



Numerical Evidence

α	S_{low} ('07)	F_{low} ('07)	FS_{low} ('10)	S_{high} ('07)	F_{high} ('07)	FS_{high} ('10)	PS (1951)	HH_{high} ('15)	HH_{low} ('15)
4	57.524	91.819	**93.340**	**102.265**	109.626	111.238	134.828	**115.166**	35.5359
6	73.668	120.369	**121.04**	130.965	**127.928**	150.04	172.665	**131.635**	59.9441
8	105.017	166.873	**167.292**	186.697	**171.833**	219.06	246.143	**175.095**	94.0015
10	145.918	224.803	**225.104**	259.409	**228.582**	308.148	342.007	**231.662**	137.03
12	195.398	292.766	**293.000**	347.374	**295.863**	415.421	457.981	**298.844**	188.844
14	253.157	370.237	**370.427**	450.058	**372.886**	540.223	593.36	**375.819**	249.353
16	319.072	456.946	**457.105**	567.239	**459.275**	682.237	747.853	**462.174**	318.502
18	393.082	552.731	**552.867**	698.812	**554.819**	841.282	921.319	**557.694**	396.247
20	475.154	657.479	**657.598**	844.718	**659.378**	1017.24	1113.68	**662.237**	482.556
22	565.25	771.108	**771.215**	1004.92	**772.855**	1210.01	1324.9	**775.702**	577.4
24	663.416	893.555	**893.651**	1179.41	**895.176**	1419.54	1554.94	**898.014**	680.754
26	769.586	1024.77	**1024.85**	1368.15	**1026.28**	1645.78	1803.79	**1029.11**	792.597
28	883.774	1164.7	**1164.78**	1571.15	**1166.13**	1888.68	2071.42	**1168.95**	912.911
30	1005.98	1313.32	**1313.39**	1788.4	**1314.66**	2148.2	2357.84	**1317.49**	1041.68
32	1136.19	1470.59	**1470.66**	2019.89	**1471.87**	2424.32	2663.04	**1474.68**	1178.88

Table: Comparison of upper and lower bounds for λ for a right triangle with sides $a = \cos\pi/\alpha, b = \sin\pi/\alpha, 1$, as a function of α: Siudeja, 2007 (S_{low}, S_{high}), Freitas, 2007 (F_{low}, F_{high}), Freitas-Siudeja, 2010 (FS_{low}, FS_{high}), Payne-Weinberger, 1960 (F_{low}), Pólya-Szego (PS) 1951, and Hasnaoui-Hermi (HH_{high} and HH_{low}, this article).

Numerical Evidence: Regular $\alpha-$polygon with n sides

n	$\alpha = 1$		$\alpha = 1.5$		$\alpha = 2$		$\alpha = 3$		$\alpha = 4$		$\alpha = 5$		$\alpha = 6$	
2	18.401	29.364	20.640	26.921	25.533	30.900	38.493	43.629	54.473	59.861	73.098	78.869	94.236	100.437
3	16.145	19.576	20.380	22.866	26.001	28.268	39.718	41.972	56.194	58.584	75.224	77.779	96.724	99.482
4	15.585	17.201	20.408	21.640	26.284	27.418	40.285	41.412	56.952	58.142	77.416	76.142	97.784	99.150
5	15.315	16.232	20.397	21.103	26.389	27.036	40.520	41.156	57.273	57.940	77.244	76.533	98.237	98.997
6	15.153	15.736	20.371	20.819	26.425	26.832	40.623	41.018	57.420	57.830	77.150	76.715	98.451	98.914
7	15.046	15.447	20.345	20.649	26.435	26.709	40.672	40.935	57.493	57.764	77.094	76.808	98.561	98.864
8	14.972	15.293	20.322	20.541	26.435	26.630	40.696	40.881	57.533	57.722	77.058	76.859	98.622	98.832
9	14.918	15.138	20.302	20.467	26.431	26.577	40.709	40.845	57.555	57.693	77.033	76.889	98.658	98.810
10	14.879	15.050	20.287	20.414	26.427	26.538	40.715	40.818	57.568	57.672	76.907	77.015	98.681	98.794
11	14.847	14.986	20.274	20.375	26.422	26.509	40.718	40.799	57.576	57.656	76.919	77.002	98.695	98.782
12	14.823	14.936	20.263	20.345	26.417	26.488	40.720	40.784	57.581	57.645	76.926	76.992	98.705	98.773
13	14.804	14.898	20.254	20.322	26.413	26.471	40.720	40.773	57.584	57.636	76.931	76.984	98.711	98.766
14	14.788	14.868	20.247	20.304	26.409	26.458	40.720	40.763	57.586	57.628	76.934	76.978	98.718	98.761
15	14.776	14.844	20.241	20.289	26.406	26.447	40.720	40.756	57.587	57.622	76.937	76.973	98.719	98.756
$j^2_{\alpha,1}\ (n \to \infty)$	14.6820		20.1907		26.3746		40.7065		57.5829		76.9389		98.7263	

Table: Upper and lower bounds for the fundamental eigenvalue of a regular α-polygon with n-sides.

Other isoperimetric results

(P_α is relative torsional rigidity, which we define in later slides) Saint Venant for P_α (H., Hasnaoui, AHP 2014)

$$P_\alpha \leq \frac{1}{\alpha+2}\left(\frac{\alpha}{(4\alpha+4)^\alpha \pi}\right)^{\frac{1}{\alpha+1}} A_\alpha^{\frac{\alpha+2}{\alpha+1}}, \tag{5}$$

Weighted reverse Hölder ineq. (H., Hasnaoui, AHP 2014)

$$\int_D u^2 dA \leq \frac{\alpha}{\pi j_{\alpha,1}^{2\alpha}} \lambda^{\alpha+2} \left(\int_D u\, h dA\right)^2. \tag{6}$$

Weighted Crooke-Sperb ineq. (H., Hasnaoui, AHP 2014)

$$\left(\int_D u\, h\, dA\right)^2 \leq \frac{2B_\alpha}{\lambda} \int_D u^2\, dA. \tag{7}$$

$$\lambda \geq \left(\frac{\pi}{2\alpha}\right)^{1/(\alpha+1)} \frac{j_{\alpha,1}^{2\alpha/(\alpha+1)}}{B_\alpha^{1/(\alpha+1)}}. \tag{8}$$

Equality in all of these inequalities is attained for the perfect sector.

Weighted reverse Hölder ineq. of Chiti type

(6) is in fact a particular case of the following theorem:
Let D be a bounded domain in the wedge $\mathcal{W}$. Let p, q be real numbers such that $q \geq p > 0$, then u satisfies the inequality

$$\left(\int_D u^q h^{2-q} dA\right)^{\frac{1}{q}} \leq K(p,q,\lambda,\alpha)\left(\int_D u^p\, h^{2-p} dA\right)^{\frac{1}{p}} \tag{9}$$

with

$$K(p,q,\lambda,\alpha) = \left(\frac{\pi}{2\alpha}\right)^{\frac{p-q}{pq}} \lambda^{(\alpha+1)\frac{q-p}{pq}} \frac{\left(\int_0^{j_{\alpha,1}} r^{(2-q)\alpha+1} J_\alpha^q(r) dr\right)^{\frac{1}{q}}}{\left(\int_0^{j_{\alpha,1}} r^{(2-p)\alpha+1} J_\alpha^p(r) dr\right)^{\frac{1}{p}}}.$$

The result is isoperimetric in the sense that equality holds if and only if D is a circular sector of angle $\frac{\pi}{\alpha}$.
Remark: Sending $q \to \infty$ and $p \to 0+$, one gets the Payne-Weinberger ineq. (4)

$$\int_D h^2 dA \geq \frac{\pi j_{\alpha,1}^{2\alpha+2}}{4\alpha(\alpha+1)\lambda^{\alpha+1}}.$$

The geometric factor B_α

Theorem: For the wedge-like membrane D defined by (1) we have

$$B_\alpha \geq \frac{\pi}{2\alpha}\left[\frac{4\alpha(\alpha+1)}{\pi}A_\alpha\right]^{\frac{\alpha}{\alpha+1}}. \qquad (10)$$

Equality holds if and only if D is a perfect sector of angle $\frac{\pi}{\alpha}$.
Also:

$$\left(\frac{\pi}{2\alpha}\right)^2\left[\frac{4\alpha(\alpha+1)}{\pi}A_\alpha\right]^{\frac{2\alpha+1}{\alpha+1}} \leq \left(\int_\Gamma h^2 d\sigma\right)^2 \leq (2\alpha+2)A_\alpha B_\alpha. \qquad (11)$$

Classical Isoperimetric Inequality:

Among all domains of a fixed perimeter, L, the circle encloses the largest area.

$$4\pi A \approx 12.5663 A \leq L^2.$$

Among all triangles of a fixed perimeter, the equilateral triangle has the largest area

$$12\sqrt{3}A \approx 20.78\, A \leq L^2.$$

Among all rectangles of a fixed perimeter, the square possesses the largest area

$$16\, A \leq L^2.$$

Semi-Circle:

$$\frac{2(2+\pi)^2}{\pi}\, A \approx 16.8297\, A \leq L^2.$$

Euclid (c. 300 B.C. "Elements, I, 35-38): "Two parallelograms, or triangles, which are on the same base, or on equal bases, and in the same parallels, remain equivalent in area no matter how far the sides between the parallels may be stretched out and increase in their length."

Zenodorus (ca. 200 BC –ca. 140 BC, now lost "On Isometric Figures"): Of all polygons of the same number of sides and equal perimeter the equilateral and equiangular polygon is the greatest in area.

$$\left(4N\tan\frac{\pi}{N}\right) A \leq L^2.$$

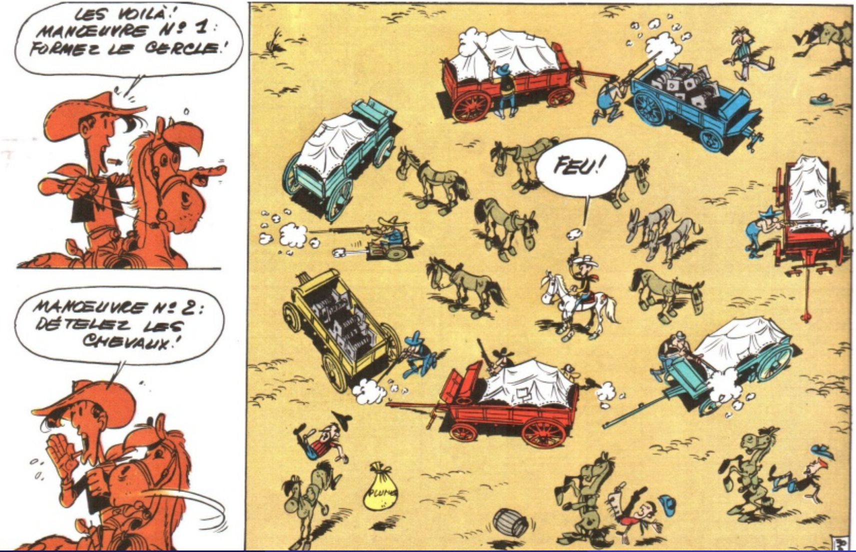
The constant has the asymptotic expansion

$$4N\tan\frac{\pi}{N} \sim 4\pi + \frac{4\pi^3}{3N^2} + \frac{8\pi^5}{15N^4} + \frac{68\pi^7}{315N^6} + \ldots$$

When $N \to \infty$, one recovers the inequality for the circle.

This is the old story of Queen Dido and the foundation of Carthage, popularized (among mathematicians) by Lord Kelvin in a public 1894 address.

Circle the Wagon: Hunker down and fight back! Bear Down!



Rayleigh-Faber-Krahn

Consider the eigenvalue problem:

$$\begin{aligned} \Delta u + \lambda u &= 0 \text{ in } D \qquad (12) \\ u &= 0 \text{ on } \partial D. \end{aligned}$$

Rayleigh's conjecture: "If the area of a membrane be given, there must evidently be some form of the boundary for which the pitch (of the principal tone) is the gravest possible, and this form can be no other than the circle."

$$\lambda \geq \frac{\pi j_{0,1}^2}{A} \approx \frac{18.1680}{A}. \qquad (13)$$

where $j_{0,1} \approx 2.4048$.
The constant appearing in (13) can be improved, if the geometry of the underlying domain is restricted.

For example among all triangles of a fixed area, the equilateral triangle maximizes the fundamental tone:

$$\lambda \geq \frac{4\pi^2}{\sqrt{3}\,A} \approx \frac{22.7929}{A} \tag{14}$$

Also among all rectangles of a fixed area, the square maximizes the fundamental tone:

$$\lambda \geq \frac{19.7392}{A}. \tag{15}$$

Incidently, the conjecture for a polygon with n sides is still open!

Wedge-Like Membrane Inequalities

Payne-Weinberger inequality for wedge-like membranes (1955): Let $D \subset \mathcal{W}_\alpha$. Then

$$\lambda \geq \lambda^* = \Big(\frac{4\alpha(\alpha+1)}{\pi} \int_D h^2(r,\theta) r\, dr\, d\theta\Big)^{\frac{-1}{\alpha+1}} j_{\alpha,1}^2 \qquad (16)$$

where $h = r^\alpha \sin \alpha\theta$. Here (r, θ) are polar coordinates taken at the apex of the wedge, and $j_{\alpha,1}$ the first zero of the Bessel function $J_\alpha(x)$. Equality holds if and only if D is a circular sector $\mathcal{W}_\alpha$.

$$\lambda(D)|D|_h^{\frac{1}{\alpha+1}} \geq \lambda(D^*)|D^*|_h^{\frac{1}{\alpha+1}} \qquad (17)$$

where D^* denotes any circular sector. Here

$$|D|_h = \int_D h^2(r,\theta) r\, dr\, d\theta.$$

This inequality improves on the Faber-Krahn inequality for certain domains (as is the case of certain triangles) and has the interpretation of being a version of Faber-Krahn in dimension $2\alpha + 2$ for axially symmetric domains (Bandle, Payne have the details).
The proof of this inequality relies on a geometric isoperimetric inequality for the quantity

$$|D|_h = A_\alpha = \int_D h^2(r, \theta) r \, dr \, d\theta$$

which is optimized for the circular sector in $\mathcal{W}_\alpha$, and a carefully crafted symmetrization argument (weighted symmetric decreasing rearrangement).

Geometric inequality for wedge-like membranes

For $D \subset \mathcal{W}_\alpha = \{(r,\theta) : 0 \leq \theta \leq \pi/\alpha\}$, $\alpha \geq 1$.

$$\left(\frac{2\alpha}{\pi} \oint_{\partial D} h^2(r,\theta) ds \right)^{(2\alpha+2)/(2\alpha+1)} \geq \frac{4\alpha(\alpha+1)}{\pi} A_\alpha$$

with equality for the perfect circular sector. This statement is equivalent to a weighted statement of the Sobolev inequality (Cabré and his students, Maderna-Salsa, Lions-Pacella, etc).

What we should retain from this presentation:

- We propose in this work a complete program for wedge-like membranes;
- One can generalize this work in the context of convex cones in higher-dimensions. (à la Lions-Pacella)
- Problem is a model for "manifolds with density" $\Delta_\mu = \frac{1}{\mu}\text{div}(\mu\nabla)$; $dV = \mu dx$; $\mu = h^2(r,\theta)$ (Grigory'an, F. Morgan)
- What we describe is a model for eigenvalue problems associated with degenerate elliptic operators.
- Renewed interest in "wedge-like membrane" isoperimetric problems (Ratzkin, Treibergs, Brock, Chiacchio, Mercaldo, etc.) with strong connections to weighted functional isoperimetric inequalities, also Cabré, etc (weighted Sobolev inequalities)

Why care?

Consider right isoceles triangle with equal sides of unit length (Payne-Weinberger)

$$\alpha = 1, \lambda_1 \geq 45.0734$$

$$\alpha = 2, \lambda_1 \geq 47.6325$$

$$\alpha = 4, \lambda_1 \geq 45.9094$$

Faber-Krahn (among all domains): $\lambda_1 \geq 36.3368$

Faber-Krahn (among all triangles): $\lambda_1 \geq \frac{4\pi^2}{\sqrt{3}\,A} \approx 45.5858$

Exact value: $\lambda_1 = 49.350625$

α	S_{low} ('07)	F_{low} ('07)	FS_{low} ('10)	S_{high} ('07)	F_{high} ('07)	FS_{high} ('10)	PS (1951)	HH_{high} ('15)	HH_{low} ('15)
4	28.7621	45.9094	**46.6702**	**51.1327**	54.8128	169.241	67.4138	57.5829	20.41

Table: Comparison of upper and lower bounds for a right isoceles triangle with unit sides.

Our work: Three Problems for wedge-like membranes

For $D \subset \mathcal{W}_\alpha$, we consider

$$\mathcal{P}_1 : \begin{cases} \Delta u + \lambda u &= 0 \quad \text{in } D \\ u &= 0 \quad \text{on } \partial D, \end{cases}$$

$$\mathcal{P}_2 : \begin{cases} -div(h^k \nabla w) &= h^k f \quad \text{in } D \\ w &= 0 \quad \text{in } \partial D \cap \mathcal{W}_\alpha, \end{cases}$$

Here $k > 0$ and $h(r, \theta) = r^\alpha \sin \alpha\theta$, as above, where the function f belongs to the weighted Lebesgue space $L^2(D, d\mu)$, and $d\mu$ is the measure defined by

$$d\mu = h^k(r, \theta)\, r dr d\theta = r^{\alpha k+1} (\sin \alpha\theta)^k \, dr d\theta. \tag{18}$$

The case $k = 2; f \equiv 1$ of $\mathcal{P}_2$

$$\mathcal{P}_3 : \begin{cases} -div(h^2 \nabla w) &= h^2 \quad \text{in } D \\ w &= 0 \quad \text{in } \partial D \cap \mathcal{W}_\alpha, \end{cases}$$

Three problems, cont'd

We claim that $\mathcal{P}_3$ is equivalent to

$$\mathcal{P}_4 : \begin{cases} -\Delta v &= h(r,\theta) \quad \text{in } D \\ v &= 0 \quad \text{in } \partial D \cap \mathcal{W}_\alpha, \end{cases}$$

To see this, let $v = hw$, in $\mathcal{P}_3$.
Relative torsional rigidity is defined via the variational formulation

$$\frac{1}{P_\alpha} = \inf_{\phi \in W_0^{1,2}(D)} \frac{\int_D |\nabla \phi|^2 \, r dr d\theta}{\left(\int_D \phi h \, r dr d\theta\right)^2}. \tag{19}$$

which is in fact equivalent to

$$\frac{1}{P_\alpha} = \inf_{\phi \in W_2(D, d\mu)} \frac{\int_D |\nabla \phi|^2 \, d\mu}{\left(\int_D \phi \, d\mu\right)^2}, \tag{20}$$

where $d\mu = h^2(r,\theta) \, r dr d\theta$.

Payne Interpretation in Fractional Weinstein Space

- Case $\alpha = 1$. In this case, D is such that $y > 0$, and $\mathcal{P}_4$ reduces to

$$\mathcal{P}_4 : \begin{cases} \Delta v + y & = & 0 & \text{in } D \\ v & = & 0 & \text{in } \partial D \cap \mathcal{W}_\alpha, \end{cases}$$

With $v = y\,w$, the problem is then

$$\mathcal{P}_4 : \begin{cases} \Delta w + \dfrac{2}{y}\dfrac{\partial w}{\partial y} & = & -1 & \text{in } D \\ w & = & 0 & \text{in } \partial D \cap \{y > 0\}, \end{cases}$$

Let the function $\Phi(x_1, x_2, x_3, x_4)$ be defined by

$$\Phi(x_1, x_2, x_3, x_4) = w(x, y) \text{ where } \quad x = x_4; \quad y = \sqrt{x_1^2 + x_2^2 + x_3^2}.$$

This function has axial symmetry with respect to the x_4-axis. It is defined on

$$D_4 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 | x = x_4, y = \sqrt{x_1^2 + x_2^2 + x_3^2}, (x, y) \in D\}$$

Payne Interpretation in Fractional Weinstein Space, Cont'd

$D_4 \subset \mathbb{R}^4$ is obtained from D via rotation around the x-axis. The function Φ satisfies

$$\Delta_4 \Phi = -1 \text{ in } D_4, \quad \Phi = 0 \text{ on } \partial D_4.$$

Note that $d = 2\alpha + 2 = 4$. Let $dV = dx_1 dx_2 dx_3 dx_4$, and

$$P = \int_{D_4} \Phi dV$$

Then

$$P_1 = \int_D v\, y\, dxdy = \int_D w\, y^2\, dxdy = \frac{1}{4\pi} \int_{D_4} \Phi dV = \frac{P}{4\pi}$$

Therefore, applying the previous inequalities for P

- Pólya-Szegő:

$$P < |D_4|\lambda^{-1}$$

So

$$\begin{aligned} P_1 &< \frac{1}{4\pi}|D_4|\lambda^{-1} \\ &= \frac{1}{4\pi}(4\pi)\left(\int_D x^2 dxdy\right)\lambda^{-1} \\ &= A_1\,\lambda^{-1} \end{aligned}$$

where $A_1 = \int_D y^2 dxdy$.

- Payne-Rayner:

$$P\lambda^3 \geq 8\frac{\pi^2}{2}j_{1,1}^2.$$

Therefore

$$P_1\lambda^3 \geq \pi j_{1,1}^2$$

Payne Interpretation in Fractional Weinstein Space, Cont'd

- Saint Venant:

$$P \leq \frac{\sqrt{2}|D_4|^{3/2}}{24\pi}$$

So

$$P_1 \leq \frac{1}{3}\left(\frac{1}{8\pi}\right)^{\frac{1}{2}} A_1^{3/2}.$$

The original interpretation in the case of λ was observed by Payne.

$$\lambda \geq \frac{1}{2}\left(\frac{\pi}{2A_1}\right)^{1/2} j_{1,1}^2$$

and is also optimized for the half-disk.

Chiti ineq. $\alpha = 1$ via Payne interpretation in Weinstein space

Remember:

$$-\Delta u = \lambda u \text{ in } D, \quad u = 0 \text{ on } \partial D. \tag{21}$$

and D is a subset of the upper half-space $\mathcal{W}_1$.

$$u = y\, w, \tag{22}$$

where w is a positive smooth function vanishing on $\partial D \cap \mathcal{W}_1$. Substitute into (21) and simplify:

$$\Delta w + \frac{2}{y}\frac{\partial w}{\partial y} = \lambda w \text{ in } D, w = 0 \text{ on } \partial D \cap \mathcal{W}_1. \tag{23}$$

Let $\Phi(x_1, x_2, x_3, x_4)$ be the function defined in $\mathbb{R}^4$ by

$$\Phi(x_1, x_2, x_3, x_4) = w(x, y) \quad \text{where} \quad x = x_1; y = \sqrt{x_2^2 + x_3^2 + x_4^2};$$

This function has axial symmetry with respect to the x_1-axis. It is defined on D_4

Weighted Chiti ineq. $\alpha = 1$ via Payne interpretation in Weinstein space, cont'd

The function Φ satisfies

$$-\Delta_4 \Phi = \lambda \Phi \text{ in } D_4, \quad \Phi = 0 \text{ on } \partial D_4.$$

For the case of the Chiti inequality for half space, we first write the statement in 4-d for D_4

$$\left[\int_{D_4} \Phi^q dV_4\right]^{\frac{1}{q}} \leq K(p,q,4)\left[\int_{D_4} \Phi^p dV_4\right]^{\frac{1}{p}}$$

for $q \geq p \geq 0$. As before, changing variables, for $p > 0$, yields

$$\left[\int_{D_4} \Phi^p dV_4\right]^{\frac{1}{p}} = (4\pi)^{\frac{1}{p}}\left[\int_D w^p y^2 dV\right]^{\frac{1}{p}}.$$

Thus

$$\left[\int_D w^q y^2 dV\right]^{\frac{1}{q}} \leq (4\pi)^{\frac{1}{p}-\frac{1}{q}} K(p,q,4)\left[\int_D w^p y^2 dV\right]^{\frac{1}{p}},$$

with equality if and only if D is a half-disk.

Payne interpretation $\alpha = 1$ for Freitas-Krejčiřik ineq.

$$\lambda \leq j^2_{d/2-1,1} \, \frac{B}{d\,|D|}. \tag{24}$$

Interpretation in the case D be the domain which lies in the half-space $\mathcal{W}_1$ such that its curved boundary Γ is some smooth polar curve:
$r = \rho(\omega), \omega \in (0, \pi)$ i.e.,

$$D = \left\{ (x, y) \in \mathcal{W}_1 \mid \quad 0 < \sqrt{x^2 + y^2} < \rho(\arccos \frac{x}{\sqrt{x^2 + y^2}}) \right\}.$$

The proof goes through two steps: (1) D_4 is a strictly star-shaped domain with respect to the origin. (2) We have the reduction between

$$B(D_4) = \int_{\partial D_4} \frac{1}{(\xi, n)} dA, \quad \text{and} \quad B_1 = \int_\Gamma \frac{y^2}{(z, \tilde{n})} d\sigma, \quad B(D_4) = 4\pi B_1$$

and, as a result

$$\lambda \leq \frac{B(D_4)}{4\,|D_4|} \, j_1^2 = \frac{B_1}{4\;A_1} \, j_1^2. \tag{25}$$

Payne Interpretation in Fractional Weinstein Space, Cont'd

- Case $\alpha = 2$. In this case, D is such that $x > 0, y > 0$, and $\mathcal{P}_4$ reduces to

$$\mathcal{P}_4 : \begin{cases} \Delta v + 2xy &= 0 \quad \text{in } D \\ v &= 0 \quad \text{in } \partial D \cap \mathcal{W}_\alpha, \end{cases}$$

With $v = 2x\,y\,w$, the problem is then

$$\mathcal{P}_4 : \begin{cases} \Delta w + \frac{2}{x}\frac{\partial w}{\partial x} + \frac{2}{y}\frac{\partial w}{\partial y} &= -1 \quad \text{in } D \\ w &= 0 \quad \text{in } \partial D \cap \{x > 0, y > 0\}, \end{cases}$$

Let the function $\Phi(x_1, x_2, x_3, y_1, y_2, y_3)$ be defined by

$$\Phi(x_1, x_2, x_3, y_1, y_2, y_3) = w(x, y)$$

with $x = \sqrt{x_1^2 + x_2^2 + x_3^2}; \quad y = \sqrt{y_1^2 + y_2^2 + y_3^2}$. This function has x and y as axes of symmetry. It is defined on

$$D_6 = \{(x_1, x_2, x_3, y_1, y_2, y_3) \in \mathbb{R}^6 | x = \sqrt{x_1^2 + x_2^2 + x_3^2}, y = \sqrt{y_1^2 + y_2^2 + y_3^2}, (x, y) \in D\}$$

obtained via two "rotations" of D around the coordinate axes.

Payne Interpretation in Fractional Weinstein Space, Cont'd

The function Φ satisfies

$$\Delta_6 \Phi = -1 \text{ in } D_6, \quad \Phi = 0 \text{ on } \partial D_6.$$

Note that $d = 2\alpha + 2 = 6$. Let $dV = dx_1 dx_2 dx_3 dy_1 dy_2 dy_3$, and

$$P = \int_{D_6} \Phi dV$$

Then

$$P_2 = 2 \int_D v\, x\, y\, dxdy = 4 \int_D w\, x^2\, y^2 dxdy = \frac{4}{(4\pi)^2} \int_{D_6} \Phi dV = \frac{P}{4\pi^2}.$$

Also

$$|D_6| = \int_{D_6} dV = (4\pi)^2 \int_D x^2 y^2\, dxdy = 4\pi^2 A_2$$

where $A_2 = 4 \int_D x^2\, y^2\, dxdy$.

Payne Interpretation in Fractional Weinstein Space, Cont'd

Therefore, applying the previous inequalities for P

- Pólya-Szegő:

$$P < |D_6|\lambda^{-1}$$

  which leads to

$$P_2 < A_2\,\lambda^{-1}.$$

- Payne-Rayner:

$$P\lambda^4 \geq 12\frac{\pi^3}{6}\,j_{2,1}^4.$$

Therefore

$$P_2\lambda^4 \geq \frac{\pi}{2}j_{2,1}^4$$

Payne Interpretation in Fractional Weinstein Space, Cont'd

- Saint Venant:

$$P \leq \frac{6^{1/3}|D_6|^{4/3}}{48\pi}$$

which simplifies as

$$P_2 \leq \frac{1}{4}\left(\frac{1}{72\pi}\right)^{\frac{1}{3}} A_2^{4/3}.$$

Again the original interpretation in the case of λ was observed by Payne

$$\lambda \geq \frac{1}{2}\left(\frac{\pi}{12A_2}\right)^{1/3} j_{2,1}^2,$$

and isoperimetry holds for the last two inequalities for the quarter disk with the same A_2 as D.

We proceed as in the above for the case of the weighted Chiti and Freitas-Krejčiřik isoperimetric inequalities in this case as well.

Pólya-Szegő for wedge-like membrane

Theorem. For a wedge-like membrane $D \subset \mathcal{W}_\alpha$

$$P_\alpha \leq |D|_h \, \lambda^{-1}$$

where

$$|D|_h = A_\alpha = \int_D h^2 dxdy$$

Proof. Start with

$$\mathcal{P}_4 : \begin{cases} -\Delta v = h(r,\theta) & \text{in } D \\ v = 0 & \text{in } \partial D \cap \mathcal{W}, \end{cases}$$

$$\begin{aligned} P_\alpha = \frac{\left(\int_D v\, h\, dxdy\right)^2}{\int_D |\nabla v|^2} &\leq \frac{\int_D v^2 \int_D h^2}{\int_D |\nabla v|^2} \\ &\leq \lambda^{-1} \, |D|_h \end{aligned}$$

Another Proof inspired by Pólya-Szegő

Since the eigenfunctions $\{u_n\}_{n=1}^{\infty}$ form an orthonormal basis of $L^2(D)$, corresponding to the eigenvalues $0 < \lambda \equiv \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_n \to \infty$, one can write

$$|D|_h = \int_D h^2\, dA = \sum_{n=1}^{\infty} \left(\int_D h\, u_n\, dA \right)^2, \tag{26}$$

and

$$P_\alpha = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \left(\int_D h\, u_n\, dA \right)^2. \tag{27}$$

The result is then immediate from the ordering of the eigenvalues, viz.

$$P_\alpha < \frac{1}{\lambda_1} \sum_{n=1}^{\infty} \left(\int_D h\, u_n\, dA \right)^2 = \frac{1}{\lambda_1} |D|_h.$$

Key: Expand $h = \sum_{n=1}^{\infty} \alpha_n\, u_n$ with $\alpha_n = \int_D h u_n\, dA$, and $v = \sum_{n=1}^{\infty} \beta_n\, u_n$, then use Plancherel-Parseval.

Payne-Rayner for wedge-like membrane

Theorem. For a wedge-like membrane $D \subset \mathcal{W}_\alpha$

$$P_\alpha \lambda^{\alpha+2} \geq \frac{\pi}{\alpha} j_\alpha^{2\alpha}$$

Proof. Start with Rayleigh-Ritz

$$\frac{1}{P_\alpha} \leq \frac{\int_D |\nabla u|^2 \, rdrd\theta}{\left(\int_D uh \, rdrd\theta\right)^2}.$$

u being the fundamental eigenfunction. Then

$$\frac{1}{P_\alpha} \leq \frac{\int_D |\nabla u|^2 \, rdrd\theta}{\int_D u^2 \, rdrd\theta} \frac{\int_D u^2 \, rdrd\theta}{\left(\int_D uh \, rdrd\theta\right)^2}.$$

Complete argument with weighted Chiti inequality.

Saint Venant for wedge-like membrane

Theorem (Hasnaoui-H. AHP 2014, and this paper) For a wedge-like membrane $D \subset \mathcal{W}_\alpha$

$$\frac{A_\alpha^2}{(2\alpha+4)B_\alpha} \leq P_\alpha \leq \frac{1}{\alpha+2}\left(\frac{\alpha\,|D|_h^{\alpha+2}}{4^\alpha(\alpha+1)^\alpha\,\pi}\right)^{1/(\alpha+1)}$$

Equality holds for the circular sector.
One can produce scale-free versions as well:

$$P_\alpha(D)|D|_h^{-\frac{2\alpha+4}{2\alpha+2}} \leq P_\alpha(D^*)|D^*|_h^{-\frac{2\alpha+4}{2\alpha+2}}. \quad (28)$$

This is a corollary (case $k=2$, $f=1$ of the following more general setting)

Rearrangement of Functions

We let f be a real-valued measurable function defined in $D \subset \mathbb{R}^n$, and let μ be its distribution function defined, for every nonnegative t, by

$$\mu(t) = |\{x \in D : |f(x)| > t\}| = \int_{\{|f(x)|>t\}} dx.$$

The *decreasing rearrangement* of f is the function f^* defined by

$$f^*(s) = \inf\{t \geq 0 : \mu(t) < s\}.$$

The function $f^\star$ defined by $f^\star(x) = f^*(C_n|x|^n)$, where $C_n = \dfrac{\pi^{n/2}}{\Gamma(n/2+1)}$, is called the *spherically symmetric-decreasing rearrangement* of f. The *spherically symmetric-increasing rearrangement* of f, denoted $f_\star$, is defined in a similar fashion. While f^* is defined on $[0, |D|]$, $f^\star$ is defined on the ball $D^\star$ centered at the origin of the same volume as D.

Flavor of the Proof of the Main Theorem

The fundamental eigenvalue λ of the wedge-like membrane D defined above satisfies the (new) inequality in red.

$$\left(\frac{4\alpha(\alpha+1)}{\pi}A_\alpha\right)^{\frac{-1}{\alpha+1}} j_{\alpha,1}^2 \leq \lambda \leq \frac{B_\alpha}{(2\alpha+2)A_\alpha} j_{\alpha,1}^2. \tag{29}$$

Equality holds if and only if D is a perfect sector S_0.
Key ingredient: In the Rayleigh-Ritz's characterization, let the test function be $f = vh$, where

$$v(r,\theta) = g\left(\frac{r}{\rho(\theta)}\right), \tag{30}$$

and g is any C^1 function such that $g(1) = 0$. This reduces the problem to a 1-d setting. (This is a "weighted proof" of the Pólya-Szegö proof, both in the case of λ and P_α.)