Singularities of the Wave Trace

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Joint work with Yves Colin de Verdière and Victor Guillemin

- Wave operator and its trace
- Length spectrum
- $\mathbb{R}^2/2\pi\mathbb{Z}^2$, disk
- Friedlander model
- Oscillatory integrals

(M,g) compact

$$(\partial_t^2 - \Delta)u = 0; \quad u_t(x,0) = 0; \quad u(x,0) = f(x)$$

$$u = \cos(t\sqrt{\Delta})f = \int_M K_t(x,y)f(y)\,dy$$

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$$K_t(x,y) = \sum_j \cos(t\lambda_j)\varphi_j(x)\varphi_j(y);$$

$$(\Delta\varphi_j = -\lambda_j^2\varphi_j; \quad \|\varphi_j\|_{L^2(M)} = 1)$$

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Theorem (Chazarain, Duistermaat-Guillemin)

$$\text{singular support}(W) = \overline{\mathcal{L}} \quad (\text{if } \partial M = \emptyset)$$

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Andersson-Melrose: $\partial M \neq \emptyset$, convex

Example 1. $P = a_{jk}\partial_j\partial_k$ **on** $\mathbb{T}^2 = \mathbb{R}^2/L$, $L = \mathbb{Z}\mathbf{v_1} + \mathbb{Z}\mathbf{v_2}$

$$\Delta e^{ix\cdot\lambda} = -H(\lambda)^2 e^{ix\cdot\lambda} \qquad (\lambda\cdot\ell \in 2\pi\mathbb{Z} \quad \forall \ell \in L)$$

$$Z(t) = \sum_{\lambda \in 2\pi L'} e^{itH(\lambda)}$$

Hamiltonian flow for $H(\xi) = \sqrt{a_{jk}\xi_j\xi_k}$ satisfies

$$(x(0), \xi(0)) = (x_0, \xi_0) \implies x(t) = x_0 + t\nabla H(\xi_0); \quad \xi(t) = \xi_0$$

For each $\ell \in L$, there is a closed geodesic with period $T = T(\ell)$

$$\boxed{T\nabla H(\xi_0) = \ell}$$

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Proof: $$\sum_{\lambda \in 2\pi L'} \varphi(\lambda) = c \sum_{\ell \in L} \hat{\varphi}(\ell), \quad \varphi(\xi) = e^{itH(\xi)}.$$

$$Z(t) = c \sum Z_\ell(t); \quad Z_\ell(t) = \int_{\mathbb{R}^2} e^{-i\xi \cdot \ell} e^{itH(\xi)} \, d\xi$$

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Stationary points

$$\nabla_\xi[-\xi \cdot \ell + tH(\xi)] = 0 \iff -\ell + t\nabla H = 0$$

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Theorem (Colin de Verdière, Guillemin, J–; 2008)

$$W \in C^\infty([2\pi, 8))$$

Today: proof using oscillatory integrals.

$$\varphi_{m,n}(re^{in\theta}) = J_n(r\rho(m,n))e^{in\theta}$$
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$\rho(m,n)$ are zeros of the Bessel function J_n:

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$$Z(t) = \sum_{m=1}^{\infty} \sum_{n\in\mathbb{Z}} e^{it\rho(m,n)}$$

Theorem (zeros of Bessel functions in transition region)

$$\rho(m,n) - n \in \Sigma^{2/3,1/3}$$
$$(\rho(m,n) - n)^{-1} \in \Sigma^{-2/3,-1/3}, \quad 1 \leq m \leq cn$$

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New/Oldest Method: Oscillatory integrals 1912; 1970s; now

$$J_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(x \sin t - nt)}\, dt$$

Friedlander Model

$$L = \frac{\partial^2}{\partial x^2} + (1+x)\frac{\partial^2}{\partial \theta^2}; \quad x \geq 0, \quad \theta \in \mathbb{R}/2\pi\mathbb{Z} = \mathbb{T}$$

$$L\psi_{m,n} = -\mu(m,n)^2\psi_{m,n}; \quad \mu(m,n)^2 = n^2 + n^{4/3}t_m$$

$$\psi_{m,n}(x,\theta) = A(n^{2/3}x - t_m)e^{in\theta}, \quad n \neq 0$$

Airy Function

$$-A''(x) + xA(x) = 0, \quad \lim_{x\to\infty} A(x) = 0$$

$$A(-t_m) = 0, \quad 0 < t_1 < t_2 < \cdots$$

Proposition $t_m \sim cm^{2/3}$.

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Proposition $t_m \sim c m^{2/3}$.

$$A(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(x\xi + \xi^3/3)}\, d\xi$$

$$A(-s^{2/3}) = \frac{s^{-1/6}}{\sqrt{\pi}} \mathrm{Re}\, \mathrm{e}^{\mathrm{i}(2\mathrm{s}/3 + \mathrm{a}(\mathrm{s}))}$$

$$a(s) \sim a_0 + a_1 s^{-1} + \cdots, \quad s \to \infty$$

$$Z_F(t) = 2 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} e^{it\mu(m,n)}$$

SYMBOL CLASS

$$\mu(m,n) - n \sim m^{2/3} n^{1/3} \in \Sigma^{2/3,1/3} \quad (m \leq cn)$$

$$|\partial_m^k \partial_n^\ell (\mu(m,n) - n)| \leq C_{k,\ell} m^{2/3-k} n^{1/3-\ell}$$

Moreover,

$$(\mu(m,n) - n)^{-1} \in \Sigma^{-2/3,-1/3} \quad (m \leq cn)$$

Proof

$$Z_{k,\ell}(t) = \int_{\mathbb{R}^2} e^{-2\pi(km+\ell n)+it\mu(m,n)} \psi_1(m,n)\, dm\, dn$$

$$e^{i(t\mu - 2\pi km - 2\pi n)} = \frac{\partial_n e^{i(t\mu - 2\pi km - 2\pi \ell n)}}{i(t\partial_n \mu - 2\pi\ell)}$$

For $\ell = 1,\ t \geq 2\pi$,

$$\psi_1(m,n)(t\partial_n\mu - 2\pi)^{-1} \in \Sigma^{-2/3,-1/3}$$

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To handle the sum over k,

$$e^{-2\pi ikm} = \frac{\partial_m e^{-2\pi ikm}}{-2\pi ik}; \quad \partial_m\mu \in \Sigma^{-1/3,1/3}$$

Uniform asymptotics as $n \to \infty$, $x \to \infty$,

$$J_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(x \sin t - nt)} \, dt$$

Lemma 1

$$\int_{\mathbb{R}} e^{i\omega t^2/2} a(y,t) \, dt = \left| \frac{2\pi}{\omega} \right|^{1/2} e^{i\sigma\pi/4} (a(y,0) + b(y,\omega))$$

with

$$b(y,\omega) \in S^{-1} \sim b_{-1}(y)\omega^{-1} + b_{-2}(y)\omega^{-2} + \cdots$$

Lemma 2

$$x^{1/3}\int_{\mathbb{R}} e^{ix(t^3/3-\alpha t)} w(\alpha,t)\,dt$$
$$= A(-\alpha x^{2/3}) b_1(\alpha, x^2) + x^{-4/3} A'(-\alpha x^{2/3}) b_2(\alpha, x^2)$$

with

$$b_1(\alpha, x^2) = w(\alpha, \sqrt{\alpha}) + a_{-1}(\alpha) x^{-2} + a_{-2}(\alpha) x^{-4} + \cdots$$

and a similar symbol property for b_2.

Lemma 3: Normal form; cubic polynomial phase

There an odd, smooth diffeo $\psi_u(\tau) = t$ so that

$$t - \sin t - ut = \tau^3/3 - \alpha(u)\tau$$

with α a smooth function satisfying $\alpha(0) = 0$, $\alpha'(0) = 2^{1/3}$.

Semiclassical microlocal analysis:

$$(\partial_t^2 - \Delta) u_h = O(h^\infty), \quad h \to 0$$

$$u_h(x,0) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{i(x-x_0)\cdot(\xi/h)} \chi(\xi) \frac{d\xi}{h^2} + O(h^\infty)$$

G. Lebeau 2006 (convex domains)
O. Ivanovici, G. Lebeau, F. Planchon 2012

$$u_h(x,t) = \sum_N u_h^N(x,t),$$

$$u_h^N(x,t) = \int_{\mathbb{R}^m} e^{i\Psi_N(x,t,s)/h} a_N(x,t,s,h)\, ds$$

($N =$ number of reflections.)

$$\varphi_\lambda(x) = e^{ix\cdot\lambda}, \quad x \in X = \mathbb{R}^2/L$$

$$\boxed{\lambda \cdot \ell \in 2\pi\mathbb{Z} \quad \forall\, \ell \in L}$$

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$$\zeta(x) = x\cdot\lambda; \quad \Lambda = \{(x,\nabla\zeta(x)) = (x,\lambda) : x \in X\} \subset T^*(X).$$

$$\iff \int_{\tilde\gamma} \alpha \in 2\pi\mathbb{Z} \quad \forall\tilde\gamma \in H_1(\Lambda,\mathbb{Z}) \qquad \alpha = \sum \xi_j dx_j$$

$$\text{Disk:} \quad P_1 = \partial_1^2 + \partial_2^2; \quad P_2 = i(x_1\partial_2 - x_2\partial_1)$$

Seek joint eigenfunctions of the form

$$P_j(a(x,h)e^{i\zeta(x)/h}) = p_j(x, d\zeta/h)(a(x,0)e^{i\zeta/h})(1+O(h))$$

$$p = (p_1, p_2) : T^*X \to \mathbb{R}^2; \quad \Lambda = p^{-1}(\lambda) = \{(x, d\zeta/h)\}$$

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Bohr-Sommerfeld condition on λ:

$$\int_\gamma \alpha \in 2\pi\mathbb{Z}, \quad \gamma \in H_1(\Lambda, \mathbb{Z})$$

Let $\cos\theta = v = n/x$ be fixed $|v| \leq c < 1$, then

$$J_n(x) = (\text{positive quantity})\mathrm{Re}\, e^{ix(\sin\theta - v\theta) + O(1)}$$

(stationary phase). Hence $x = \rho(m,n)$ solves

$$\sqrt{\rho^2 - n^2} - n\cos^{-1}(n/\rho) = \pi m + O(1)$$

The Bohr-Sommerfeld property also determines the principal symbol of $\rho(m,n)$.

On the Lagrangian torus $L = x\eta - y\xi = L_0$; $H = \sqrt{\xi^2 + \eta^2} = H_0$, the action integrals are

$$I_1 = \int_{\gamma_2} \xi dx + \eta dy = 2L_0 \cos^{-1}(L_0/H_0) + 2\sqrt{H_0^2 - L_0^2}$$

$$I_2 = \int_{\gamma_1} \xi dx + \eta dy = \pm 2\pi L_0$$

Setting $(I_1, I_2) = 2\pi(m, n)$ gives the same equation for H_0 as we found for ρ.

Similarly, in the Friedlander model, the action integrals on the torus

$$H = \sqrt{\xi^2 + (1+x)\eta^2} = H_0, \quad \eta = \eta_0$$

are

$$I_1 = (4/3)\xi_0^3/\eta_0^2; \quad I_2 = 2\pi\eta_0$$

With H_0 replaced by $\mu = \mu(m,n)$, the equations $(I_1, I_2) = 2\pi(m,n)$ yield

$$\mu^2 = n^2 + (3\pi/2)^{2/3} m^{2/3} n^{4/3}$$